Magic Coefficients

We have stated that an optimal solution to the dual has

$$y_i^* = -\text{coefficient of the } i\text{th slack of the primal.}$$

To see the origin of these coefficients, we revisit our proof of Strong Duality.

Let A be an $m \times n$ matrix. We obtain the Revised Simplex Formulas (our dictionaries!) by first writing $A\mathbf{x} \leq \mathbf{b}$ as $[AI]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_S\end{bmatrix} = \mathbf{b}$ where we have n original variables (typically our decision variables) and m slack variables. Then considering the variables split into the m basic variables and the n non basic variables for some column basis B of $[AI]$ we obtain the Revised Simplex Formulas.

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N$$

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}A_N)\mathbf{x}_N$$

Assume that the Simplex Method has pivoted to an optimal solution given by a basis B. We assert

$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}A_N \leq \mathbf{0}.$$

We can readily assert that

$$\mathbf{c}_B^T - \mathbf{c}_B^T B^{-1}B \leq \mathbf{0}$$

since of course $\mathbf{c}_B^T - \mathbf{c}_B^T B^{-1}B = \mathbf{0}$. But now we have a symmetric expression for all variables, namely for all variables x_i

$$c_i - \mathbf{c}_B^T B^{-1}A_i \leq 0$$

where A_i denotes the column of $[AI]$ indexed by x_i. We regroup the variables into the original variables $\mathbf{x}$ and the slack variables $\mathbf{x}_S$ to obtain a new expression for z:

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}^T - \mathbf{c}_B^T B^{-1}A)\mathbf{x} + (\mathbf{0}^T - \mathbf{c}_B^T B^{-1}I)\mathbf{x}_S$$

Thus the coefficients of $\mathbf{x}_S$ in the final z-row are $\mathbf{0}^T - \mathbf{c}_B^T B^{-1}I = -\mathbf{y}^T$ when we have chosen $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$. Hence we have verified that the negatives of the coefficients of the slacks are the dual variables. I'll leave this as an exercise to show that if Phase One would terminate with $\max w < 0$ then the magic coefficients provide a proof of infeasibility of the inequalities.