Math 340 Dual Simplex resulting in infeasibility

Consider a primal

\[
\begin{align*}
\text{max} \quad & \mathbf{c} \cdot \mathbf{x} \\
\text{subject to} \quad & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{align*}
\]

If we have a dictionary with all the coefficients in the \(z\) row are negative (namely \(\mathbf{c}_N - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_N \leq \mathbf{0}^T\) then we can call this dual feasible since \(\mathbf{c}^T \mathbf{B}^{-1} \) would be a feasible solution to the dual:

\[
\begin{align*}
\text{min} \quad & \mathbf{b} \cdot \mathbf{y} \\
\text{subject to} \quad & \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \\
& \mathbf{y} \geq \mathbf{0}
\end{align*}
\]

If we start with a dictionary (for the primal) that is infeasible (namely \(\mathbf{B}^{-1} \mathbf{b} \not\geq \mathbf{0}\)) which has all the coefficients in the \(z\) row being negative then we can proceed with the Dual Simplex algorithm. The following example gives one way that this could happen but you imagine that this could occur in a sensitivity analysis problem using the dual simplex.

\[
\begin{align*}
\text{max} \quad & -3x_1 - x_2 \\
& 2x_1 + 2x_2 \leq 1 \\
& -2x_1 - x_2 \leq -2 \\
& 4x_1 + 3x_2 \leq 1 \\
\end{align*}
\]

We have our first dictionary

\[
\begin{align*}
x_3 &= 1 - 2x_1 - 2x_2 \\
x_4 &= -2 + 2x_1 + x_2 \\
x_5 &= 1 - 4x_1 - 3x_2 \\
z &= -3x_1 - x_2
\end{align*}
\]

Rather than introduce \(x_0\) and use our two phase method, we are able to embark directly on our dual simplex method. We choose \(x_4\) to leave and then (in order to preserve dual feasibility) we choose \(x_2\) as the entering variable. We obtain the following dictionary:

\[
\begin{align*}
x_3 &= -3 + 2x_1 - 2x_4 \\
x_2 &= 2 - 2x_1 + x_4 \\
x_5 &= -5 + 2x_1 - 3x_4 \\
z &= -2 - x_1 - x_4
\end{align*}
\]

Note that we have made progress (we have a better dual solution with a smaller objective function value in the dual of -2 rather than 0). We choose \(x_5\) to leave (greedily choosing the 'largest' negative coefficient) and then (in order to preserve dual feasibility) we choose \(x_1\) as the entering variable. We obtain the following dictionary:

\[
\begin{align*}
x_3 &= 2 + x_5 + x_4 \\
x_2 &= -3 - x_5 - 2x_4 \\
x_1 &= 5/2 + (1/2)x_5 + (3/2)x_4 \\
z &= -9/2 - (1/2)x_5 - (5/2)x_4
\end{align*}
\]

Again we have made progress finding a dual solution of value -9/2. We would choose \(x_2\) to leave but we are unable to find an entering variable since \((-1/2) - (5/2)\) + \(\lambda\)\((-1 - 2\) for all \(\lambda \geq 0\).
So we guess that the dual is unbounded but how can we see this? A solution which is somewhat wishful thinking is taking the current dual solution $y = (0, 5/2, 1/2)$ (obtained as $c^T B^{-1}$) which is readily obtained as the coefficients of the slack variables. Now why not add $t$ times the same coefficients from the row for $x_2$, namely $z = (0, 2, 1)$ to obtain a solution $y + tz = (0, 5/2 + 2t, 1/2 + t)$ with objective function value $-9/2 - 3t$ which shows the dual is unbounded. This wishful thinking works and you can verify that I have a parametric set of feasible dual solutions whose objective function, in the dual, goes to $-\infty$. Below I make explicit the reason why this works.

Now we have reached a place where we have a potential leaving variable but no entering variable. Imagine in general that we are doing the dual simplex method and we have $x_k$ leaving. Let $[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]$ denote the $m \times 1$ vector with a 1 in the column corresponding to $x_k$. Thus

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} b < 0$$

since the constant entry must be zero in the row corresponding to $x_k$.

If we are unable to determine an entering variable then that is because

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} A_N \leq 0$$

namely the entries in the row corresponding to $x_k$ must all be negative and the entries in that row are the row of $-B^{-1} A_N$.

Now we do the standard trickery (as done in the proof of Strong Duality). We have

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} B \leq 0^T$$

and so for any variable $x_i$ we have

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} A_i \leq 0.$$

Now regroup the variables by original variables and slack variables and we obtain

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} A \leq 0^T$$

and

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} I \leq 0^T$$

If we set $z^T = [0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1}$ then we discover that

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} A \leq 0$$

implies $z^T A \geq 0^T$ which is $A^T z \geq 0$ and

$$[0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] B^{-1} I \leq 0^T$$

yields $z^T \geq 0^T$ and so $z \geq 0$.

This is exactly what we need to have the dual be unbounded (towards $-\infty$). Assume $y$ is a dual solution. The $y + tz$ is also a dual feasible solution and, since $b \cdot z = z^T b < 0$, we have

$$b \cdot (y + tz) = b \cdot y + t b \cdot z$$

and so $\lim_{t \to \infty} b \cdot (y + tz) = -\infty$. 