MATH 340 Example of Degeneracy in pivoting process. 

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We give an example of an LP where we are forced to do what we call degenerate pivots. We still obtain an optimal solution(s). Degenerate pivots can result in cycling and you should read Chvatal’s example.

We considered the following LP in standard inequality form

\[
\begin{align*}
\text{max} & \quad 2x_1 + 2x_2 + x_3 \\
2x_1 & \quad + x_3 \quad \leq \quad 4 & x_1, x_2, x_3, x_4 \geq 0 \\
x_1 & \quad + x_2 \quad \leq \quad 1 \\
x_1 & \quad + x_3 \quad \leq \quad 1 \\
\end{align*}
\]

We add slack variables \(x_4, x_5, x_6\) corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables \(x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\). We form our first dictionary

\[
\begin{align*}
x_4 &= 4 - 2x_1 - x_3 \\
x_5 &= 1 - x_1 - x_2 \\
x_6 &= 1 - x_1 - x_3 \\
z &= 2x_1 + 2x_2 + x_3 \\
\end{align*}
\]

It is traditional to use \(z\) for the objective function. There is an obvious solution to these 4 equations, namely \(x_4 = 4, x_5 = 1, x_6 = 1\) and \(x_1 = x_2 = x_3 = 0\) with \(z = 0\). (this is called a basic feasible solution)

We now use Anstee’s rule trying to increase a variable from 0 in the current obvious solution so we greedily choose \(x_1\) to increase and hence enter. We leave \(x_2 = x_3 = x_4 = 0\). The choice of \(x_1\) as the variable with the largest coefficient in dictionary expression for \(z\) (and in the case of ties choosing the variable of smallest subscript) is called Anstee’s Rule in this course. If there is a tie we choose the variable with the smallest subscript.

\[
\begin{align*}
x_4 &= 4 - 2x_1 \\
x_5 &= 1 - x_1 \\
x_6 &= 1 - x_1 \\
z &= 4x_1 \\
\end{align*}
\]

We deduce that \(x_1\) can be increased to 1 while decreasing \(x_5\) and \(x_6\) to 0. We obtain a new dictionary by having \(x_1\) only appear on the left and \(x_5\) is now on the right of the equation signs. We choose \(x_5\) to leave by Anstee’s rule since there is a tie for the leaving variable and we choose the one with the smallest subscript.

\[
\begin{align*}
x_4 &= 2 + 2x_5 + 2x_2 - x_3 \\
x_1 &= 1 - x_5 - x_2 \\
x_6 &= 0 + x_5 + x_2 - x_3 \\
z &= 2 - 2x_5 + x_3 \\
\end{align*}
\]

This dictionary yields a basic solution \(x_4 = 2, x_1 = 1, x_6 = 0\) and \(x_5 = x_2 = x_3 = 0\) with \(z = 2\). One of our basic variable, \(x_6\), is 0. Such a basic solution is called a degenerate basic solution.

By Anstee’s rule we would wish to increase \(x_3\) leaving \(x_5 = x_2 = x_3 = 0\). We decide we can increase \(x_3\) from 0 to 0 while driving \(x_6\) from 0 down to 0. Perhaps this is a bit difficult to understand but this is the wording we want to use to mimic our more standard pivots. We call this a degenerate pivot because no variable will change value in the basic solution. The new dictionary with \(x_3\) entering and \(x_6\) leaving is:
\[
\begin{align*}
  x_4 &= 2 + x_5 + x_2 + x_6 \\
  x_1 &= 1 - x_5 - x_2 \\
  x_3 &= 0 + x_5 + x_2 - x_6 \\
  z &= 2 - x_5 + x_2 - x_6
\end{align*}
\]

Again, the current basic solution is degenerate (since \(x_3 = 0\)) and the basic solution is unchanged but note that the basis has changed which is significant. We now have \(x_2\) enter and then \(x_1\) leaves in a non-degenerate pivot (the basic solution changes). Of course we would be equally happy if the pivot was degenerate but resulted in a dictionary that reveals the basic solution is optimal.

\[
\begin{align*}
  x_4 &= 3 - x_1 + x_6 \\
  x_2 &= 1 - x_5 - x_1 \\
  x_3 &= 1 - x_1 - x_6 \\
  z &= 3 - 2x_5 - x_1 - x_6
\end{align*}
\]

The basic solution is now \(x_4 = 3\), \(x_2 = 1\) and \(x_3 = 1\) with \(x_5 = x_1 = x_6 = 0\) and \(z = 3\). We see this is an optimal solution. This example has us do a degenerate pivot but somehow, from the new basis \(\{x_1, x_3\}\), we can now see a beneficial pivot (with \(z\) increasing).

Degeneracy is relatively common in LP’s and you proceed as we have described. It is possible to enter a cycle (one example is posted) but this is extremely rare in practice; it essentially never happens so in practice you would not dwell on the possibility. But in order to show the termination of the Simplex algorithm, we need to have a way to avoid cycling.