Math 340  Cones in Linear Programming.

The book has very little on this, so here are a few details written out. Given a set \( V = \{v_1, v_2, \ldots, v_k\} \) of \( k \) vectors, we define the cone generated by \( V \) to be all positive linear combinations of the vectors in \( V \), namely

\[
\{a_1v_1 + a_2v_2 + \cdots + a_kv_k \mid a_1, a_2, \ldots, a_k \geq 0\}.
\]

If we form a matrix \([v_1v_2\cdots v_k]\), then the cone generated by \( V \) is \( \{[v_1v_2\cdots v_k]z \mid z \geq 0\} \). We see Linear Programming arising in this definition although for us the matrix \([v_1v_2\cdots v_k]\) becomes \( A^T \).

Consider the primal/dual pair:

\[
\begin{align*}
\text{max} \quad c \cdot x & \quad \text{min} \quad b \cdot y \\
\text{primal:} \quad Ax & \leq b & \text{dual:} \quad A^T y = c \quad y \geq 0 \\
\text{x free} & & \\
\end{align*}
\]

The main result is the following:

Assume \( x \) is a feasible solution to the primal. Then \( x \) is an optimal solution to the primal if and only if the gradient of the objective function (namely \( c \)) is contained in the cone generated by the gradients of the active constraints at \( x \).

We have a few more terms to define. The gradient of a linear function \( a_1x_1 + a_2x_2 + \cdots + a_nx_n \) is the vector \((a_1, a_2, \ldots, a_n)^T\). For the objective function we get \( c \). For the \( i \)th constraint we get the \( i \)th row of \( A \). An active constraint at \( x \) is one with zero slack, namely it is satisfied with equality. An non-active constraint at \( x \) is one with non zero slack, namely it is satisfied with strict inequality.

The proof of the theorem now readily follows from our duality theorems. If \( x \) is optimal then, by Strong Duality, there is an optimal dual solution \( y \) hence \( A^T y = c, y \geq 0 \). Thus \( c \) is a positive linear combination of the gradients of the constraints. Then, using the Theorem of Complementary Slackness, we deduce that \( y_i \) is zero for the constraints are not active. Thus \( c \) is a positive linear combination of the gradients of the active constraints, i.e. \( c \) is in the cone generated by the gradients of the active constraints.

Our example considered the linear system

\[
\begin{align*}
\text{max} \quad 2x_1 & + x_2 \\
-x_1 & + x_2 \leq 2 \\
x_1 & + x_2 \leq 10 \\
-x_1 & \leq 0 \\
-x_2 & \leq 0 \\
\end{align*}
\]

There are four 'corners', \( P_1 = (0,0), P_2 = (0,2), P_3 = (4,6), P_4 = (10,0) \).
The active constraints at $P_1$ are the third and fourth constraints and hence the cone generated by the active constraints is \( \{a_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mid a_1, a_2 \geq 0 \} \).

The active constraints at $P_2$ are the first and third constraints and hence the cone generated by the active constraints is \( \{a_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mid a_1, a_2 \geq 0 \} \).

The active constraints at $P_3$ are the first and second constraints and hence the cone generated by the active constraints is \( \{a_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid a_1, a_2 \geq 0 \} \).

The active constraints at $P_4$ are the second and fourth constraints and hence the cone generated by the active constraints is \( \{a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mid a_1, a_2 \geq 0 \} \).

In our case $c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and we find that \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) and so $c$ is in the cone generated by the active constraints at $P_4$. We deduce that $P_4$ is optimal without any further work.

We do not propose this as good algorithm for finding an optimal solution although the ‘active set approach’ does yield effective algorithms for linear and non-linear programming.

The idea we wish to explore is parametric programming when for example the objective function is given as a function of a parameter $p$:

\[ c = \begin{pmatrix} 2 + 2p \\ 1 - p \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + p \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \]

The 4 cones all fit together, at their tips, to form all of $\mathbb{R}^2$. The following picture may be helpful.

You might consider how this would work for a general (convex) polygon arising from inequalities particularly in 2 or 3 variables where we can draw pictures. We discover that $c$ is in the cone for
$P_2$ for $p \leq -3$, and $c$ is in the cone for $P_3$ for $-3 \leq p \leq -1/3$, and $c$ is in the cone for $P_4$ for $-1/3 \leq p$. Thus, for example, when $p \leq -3$, the optimal solution is $x_1 = 0, x_2 = 2$ and so the value of the objective function is $c \cdot x = \begin{pmatrix} 2 + 2p \\ 1 - p \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2 - 2p$. Our complete parametric answer is:

$$z = \begin{cases} 2 - 2p & \text{for } p \leq -3 \\ 14 + 2p & \text{for } -3 \leq p \leq -1/3 \\ 20 + 20p & \text{for } -1/3 \leq p \end{cases}$$

Note the form of the answer. The objective function $z$ will be a piecewise linear concave continuous function of $p$. 