I will now give an example of the two phase method that we did in class. You can also witness examples of the two phase method in the practice for quiz2.

Maximize \(-x_1 + 3x_2 + x_3\)

\[
2x_1 + x_2 \leq -1 \quad x_1, x_2, x_3 \geq 0
\]

\[-2x_1 + x_3 \leq -2
\]

\[2x_2 + 2x_3 \leq -1\]

Solution:

Phase One: We might try to write

\[
x_4 = -1 + x_1 + x_3
\]

\[
x_5 = -2 + x_1 + x_2
\]

\[
x_6 = -1 + x_2 - x_3
\]

\[
z = -2x_1 - x_2 - x_3
\]

But the associated ‘obvious’ solution (the basic solution associated with dictionary) has \(x_4 = -1\), \(x_5 = -2\) and \(x_6 = -1\) so it is not feasible. While we could pivot it would be hopeless to describe the pivot rules for this. Recall we choose the leaving variable to preserve feasibility but that won’t work if you start with an infeasible solution.

Instead we add in the artificial variable \(x_0\) to make this work:

\[
x_4 = -1 + x_1 + x_3 + x_0
\]

\[
x_5 = -2 + x_1 + x_2 + x_0
\]

\[
x_6 = -1 + x_2 - x_3 + x_0
\]

\[
w = -x_0
\]

The new objective function is \(\text{max} -x_0\) and so it attempts to drive \(x_0\) to 0. We can take \(x_0\) initially large to find a feasible solution

\[
x_4 = -1 + x_0 \geq 0 \text{ so } x_0 \geq 1
\]

\[
x_5 = -2 + x_0 \geq 0 \text{ so } x_0 \geq 2
\]

\[
x_6 = -1 + x_0 \geq 0 \text{ so } x_0 \geq 1.
\]

We conclude that \(x_0 = 2\) drives \(x_5\) to 0 while having \(x_4 = 1 \geq 0\) and \(x_6 = 1 \geq 0\). So we choose \(x_5\) to leave (it is the last variable driven to 0 as we increase \(x_0\)). These are not the usual rules and so we call this the \textit{special pivot to feasibility}.

\(x_0\) enters and \(x_5\) leaves (Special pivot to feasibility)

\[
x_4 = 1 -x_2 + x_3 + x_5
\]

\[
x_0 = 2 -x_1 - x_2 + x_5
\]

\[
x_6 = 1 -x_1 - x_3 + x_5
\]

\[
w = -2 + x_1 + x_2 - x_3
\]

This is a traditional dictionary and we now attempt to pivot to drive \(x_0\) to 0 at which point we will delete it. There is a tie for the entering variable between \(x_1\) and \(x_2\). \textbf{Anstee’s Rule} asks you to choose the smallest subscript in the event of ties. So \(x_1\) enters and \(x_6\) leaves.

\[
x_4 = 1 -x_2 + x_3 + x_5
\]

\[
x_0 = 1 +x_6 - x_2 + x_3
\]

\[
x_1 = 1 -x_6 - x_3 + x_5
\]

\[
w = -1 -x_6 + x_2 - x_3
\]
We have made progress ($x_0$ is now just 1) but carry on. $x_2$ enters. There is a tie for the leaving variable between $x_0$ and $x_4$. **Anstee’s Rule** asks you to choose the smallest subscript in the event of ties. The choice $x_0$ now becomes clear. When $x_0$ is driven to 0 in the pivoting process it will leave the basis. So $x_2$ enters and $x_0$ leaves.

$$
\begin{align*}
\begin{array}{cccc}
x_4 &= 0 & -x_6 & +x_0 & +x_5 \\
x_2 &= 1 & +x_6 & -x_0 & +x_3 \\
x_1 &= 1 & -x_6 & -x_3 & +x_5 \\
w &= 0 & & -x_0 \\
\end{array}
\end{align*}
$$

We may now delete $x_0$ and $w$ (note that we want $x_0$ to be on the left side, one of the non basic variables):

$$
\begin{align*}
\begin{array}{c}
x_4 &= 0 & -x_6 & +x_5 \\
x_2 &= 1 & +x_6 & +x_3 \\
x_1 &= 1 & -x_6 & -x_3 & +x_5 \\
\end{array}
\end{align*}
$$

This finishes Phase one. Two pivots and we have driven $x_0$ to 0 and so have a feasible dictionary to begin Phase Two. We must reintroduce $z = -2x_1 - x_2 - x_3$ to begin Phase Two but in the form that it is written in terms of the non basic variables $x_3, x_5, x_6$. To do so we need substitute for $x_1$ and $x_2$ in this simple case

$$
\begin{align*}
z &= -2(1 - x_6 - x_3 + x_5) - (1 + x_6 + x_3) - x_3 = -3 + x_6 - 2x_5 
\end{align*}
$$
yielding the dictionary

$$
\begin{align*}
\begin{array}{ccccc}
x_4 &= 0 & -x_6 & +x_5 \\
x_2 &= 1 & +x_6 & +x_3 \\
x_1 &= 1 & -x_6 & -x_3 & +x_5 \\
z &= -3 & +x_6 & -2x_5 \\
\end{array}
\end{align*}
$$

We now apply our standard pivot rules again and choose $x_6$ to enter and have $x_4$ leave. Well this is not entirely what we are used to doing. We are increasing $x_6$ from 0 to 0 while driving $x_4$ from 0 down to 0. This is called a degenerate pivot. We obtain the following dictionary.

$$
\begin{align*}
\begin{array}{ccccc}
x_6 &= 0 & -x_4 & +x_5 \\
x_2 &= 1 & -x_4 & +x_3 & +x_5 \\
x_1 &= 1 & +x_4 & -x_3 \\
z &= -3 & -x_4 & -x_5 \\
\end{array}
\end{align*}
$$

Now the actual basic solution has not changed. It is $(1,1,0,0,0,0)$ with $z = -3$ in both cases. But the set of basic variables has changed and moreover it is now clear that we are at optimality. This sort of thing occurs in Mathematics, to see an optimal solution you often have to transform the problem.