1. Give an example of a primal LP which is infeasible while simultaneously its dual LP is infeasible.

To obtain a primal/dual pair that are both infeasible, we can try making a primal whose dual has the same infeasible constraints. A $2 \times 2$ example suffices.

**primal:**

$$\begin{align*}
\text{max} & \quad x_1 - 2x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x \geq 0
\end{align*}$$

**dual:**

$$\begin{align*}
\text{min} & \quad y_1 - 2y_2 \\
\text{subject to} & \quad y_1 - y_2 \leq 1 \\
& \quad -y_1 + y_2 \leq -2 \\
& \quad y \geq 0
\end{align*}$$

2. (from an old exam) If you are given an optimal primal solution $x^*$ to an LP and you wish to deduce an optimal dual solution $y^*$, then you might try to determine $y^*$ using:

(1) Complementary Slackness of $y^*$ with $x^*$.

(2) $y^*$ satisfies constraints (including positivity constraints if any) in dual, i.e. $y^*$ is a feasible solution of the dual.

Many of our examples in class and quizzes yielded unique optimal $y^*$ but in general there may be many optimal dual solutions. Are all possible $y^*$ satisfying (1),(2) optimal to the dual? Is every optimal dual solution $y^*$ determined as a solution to (1),(2)? Explain.

The best way to tackle this question is to write down the Theorem of Complementary Slackness.

All $y^*$ satisfying (1),(2) are feasible to the dual by (2) and then since $x^*$ is feasible and (1) holds (which is complementary slackness) and so by the Complementary Slackness Theorem, we deduce $y^*$ is optimal to the dual (and $x^*$ is optimal to the primal).

Every optimal dual solution $y^*$ is feasible and hence satisfies (2). Also by the other direction of the Complementary Slackness Theorem, since $x^*$ and $y^*$ are optimal to their respective LP’s, we have that (1) holds.

This question is in essence asking for a restatement of the Complementary Slackness Theorem.

3. a) I used $d_1d_2d_3d_4d_5 = 54321$ and obtained

$$\begin{align*}
\text{max} & \quad c_1x_1 + 7x_2 + 7x_3 + 11.1x_4 \\
\text{subject to} & \quad 15x_1 + 4.5x_2 + 1.2x_3 + 9x_4 \leq 100 \\
& \quad 14x_1 + 4x_2 + 1x_3 + 8.2x_4 \leq 100 \\
& \quad 13x_1 + 3x_2 + 3x_3 + 3x_4 \leq 100 \\
& \quad 12x_1 + 2x_2 + 4x_3 + x_4 \leq 100 \\
& \quad 11x_1 + x_2 + 5x_3 + x_4 \leq 100 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}$$

There were three intervals:

$c_1 \in (-\infty, 31.244]$ gives solution $(0, 15.55, 16.66, 1.11)$ with $z = 237.88$

$c_1 \in [31.244, 87.5]$ gives solution $(6.149, 0, 6.472, 0)$ with $z = 6.149c_1 + 45.304$

$c_1 \in [87.5, +\infty)$ gives solution $(6.666, 0, 0, 0)$ with $z = 6.666c_1$

You may wish to note that since we are only changing $c_1$ and we look at intervals in which the optimal basis is unchanged, then the optimal solution $B^{-1}b$ stays unchanged in that interval.
Thus the slope of the line in an interval is the value of $x_1$. The extent of the interval is found by checking the objective function ranging for the coefficient of $x_1$ ($c_1$) which gives the interval (around the current value of $c_1$) for which the current basis remains an optimal basis and so for which the solution remains unchanged.

One lucky (?) student had a degeneracy arise ($d_1d_2d_3d_4d_5 = 41411$) and so there were two different bases and two associated intervals for which the value of $x_1$ and the other variables were the same in both and hence in the graph, the intervals could be combined into one interval.

b) To show that the (piecewise linear) curve is concave upwards it suffices to show that for each interval where we have computed $z = x_1c_1 + \text{constant}$ for $a \leq c_1 \leq b$ then $z \geq x_1c_1 + \text{constant}$ for all values of $c_1$. This is easily seen to be true since the feasible solution which achieves $z = x_1c_1 + \text{constant}$ is a feasible solution regardless of $c_1$ and so we have a feasible solution to our LP of value $x_1c_1 + \text{constant}$ for all values of $c_1$ and hence the optimal value of the objective function is at least this big.

4.
a) Show there is an $x \geq 0$ with $Ax < 0$ if and only if there is an $x \geq 0$ with $Ax \leq -1$.

Note: we use the definition $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_1 < y_1, x_2 < y_2, \ldots$ and $x_n < y_n$. This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol $>$ would generally mean $\geq$ and $\neq$ but this is not true for matrices or vectors. A vector $x$ might satisfy $x \geq 0$ and also $x \neq 0$. If $x \geq 0$ and yet $x$ has some 0 entries then $x > 0$. This note is just to explain things to you. The notation ‘$>$’ is different for matrices and vectors than you might first think. What follows is the proof of the question a):

If there is an $x \geq 0$ with $Ax \leq -1$, then that $x$ satisfies with $Ax \leq -1 < 0$.

If there is an $x \geq 0$ with $Ax < 0$ then assume such an $x$ exists with $Ax = (-a_1, -a_2, \ldots, -a_m)^T$. Let $a = \min\{a_1, a_2, a_3, \ldots, a_m\}$. Thus $a > 0$. Then $A(\frac{1}{a}x) = (-a_1/a, -a_2/a, \ldots, -a_m/a)^T \leq -1$ and $\frac{1}{a}x \geq 0$.

This result is particularly interesting to see a strict inequality appear in an LP.

b) We set up a primal dual pair.

<table>
<thead>
<tr>
<th>primal P:</th>
<th>max $0 \cdot x$</th>
<th>dual D:</th>
<th>min $-1 \cdot y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ax \leq -1$</td>
<td>$A^T y \geq 0$</td>
<td>$y \geq 0$</td>
<td>$z \geq 0$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$A^T y \geq 0$</td>
<td>$z \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Note that we have not introduced spurious ideas such as $Ax < 0$ or $y \neq 0$. They don’t fit into an LP. We have two statements:

i) there exists an $x \geq 0$ with $Ax < 0$,

ii) there exists $y \geq 0, z \geq 0$ with $A^T y \geq 0$ and $y \neq 0$

Our primal $P$ is bounded (value of objective function at most 0) and so there are two cases by Fundamental Theorem of LP.

Case 1. Assume that the primal is infeasible.
The dual is feasible \((y = 0 \text{ works})\) so by the Fundamental Theorem of Linear Programming, the dual is either unbounded or has an optimal solution. But if the dual has an optimal solution, then by Strong Duality, we deduce that the primal has an optimal solution \(x\) which is feasible which is a contradiction. Thus the dual is unbounded and so we can find a feasible \(y\) (i.e. \(A^Ty \geq 0\)) with \(-1 \cdot y < 0\) and so \(y \neq 0\) and hence ii) holds.

The primal is infeasible and so by a) we have that there can be no \(x \geq 0\) with \(Ax \leq -1\) which means by our argument in part a) that there is no \(x \geq 0\) with \(Ax < 0\) and we conclude i) doesn’t hold.

Case 2. Assume the primal has an optimal solution \(x^*\).

We apply a) again to note that if there is a feasible solution to the primal then there is an \(x \geq 0\) with \(Ax < 0\). Thus i) holds

Also, by Weak Duality, any feasible solution to the dual (i.e. any \(y\) with \(A^Ty \geq 0, y \geq 0\)) has \(-1 \cdot y \geq 0 \cdot x^* = 0\). But \(-1 \cdot y \geq 0\) and \(y \geq 0\) implies that \(y = 0\) and so ii) doesn’t hold. Since Case 1 and 2 exhaust all possibilities we know that either i) or ii) holds but not both.

5. (from an old exam) We seek a minimum cost diet selected from the following three foods.

<table>
<thead>
<tr>
<th></th>
<th>food 1</th>
<th>food 2</th>
<th>food 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>vitamins</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>calories</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>minimum</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cost $/100gm</td>
<td>3.00</td>
<td>5.00</td>
<td>8.00</td>
</tr>
</tbody>
</table>

We require a diet that has at least 760 units of vitamins and at least 3500 calories. The minimums are stated in units of 100gms. We let the variable food\(i\) refer to the amount of food \(i\) purchased in units of 100gms.

a) There is a special on food 2 reducing the price to $4.10/100gms. This would not change the your purchase strategy because the ‘current basis remains optimal’ and so the current basic feasible solution \(B^{-1}b\) remains fixed even with a drop of $.91/100 gms. A price reduction to $3.10 falls outside the range and one imagines the purchase strategy would change.

b) The marginal cost of 10 units of vitamins is \(10 \times \cdot 2222\) which is $2.22. The chosen diet as a whole has 760 units of vitamins and costs $178.60 and so the dollar cost of the whole diet per 10 units of vitamins obtained is $178/76 = $2.35. The marginal cost is cheaper (which is not that surprising since the diet also has other constraints).

c) Integrality can be important in diet problems such as this if the foods come in integer amounts (e.g. apples, restaurant meals) but many foods are available in continuous amounts (vegetables by the kilo, bulk food bins) and anyway can be divided at home into appropriate amounts. I expected some discussion of the possibility that the variables for the foods were integral.

d) A linear inequality that expresses the requirement that at least 20% of the weight of the purchased diet comes from food 2 is

\[
\text{food2} \geq .2(\text{food1} + \text{food2} + \text{food3})
\]

which in LINDO input would become \(.2\text{food1} - .8\text{food2} + .2\text{food3} \leq 0\).