1. Consider an LP which has as its first constraint

\[-x_1 + x_2 - x_3 - x_4 \leq 1.\]

Assume that you know (based on other constraints) that \(x_2 = 0\). Show that the dual variable \(y_1\) associated with the first constraint is zero.

2. A question using LINDO (or other software). Consider the following LP:

\[
\begin{align*}
\text{max} & \quad c_1x_1 + 7x_2 + 7x_3 + 11.1x_4 \\
(10 + d_1)x_1 + 4.5x_2 + 1.2x_3 + 9x_4 & \leq 100 \\
(10 + d_2)x_1 + 4x_2 + x_3 + 8.2x_4 & \leq 100 \\
(10 + d_3)x_1 + 3x_2 + 3x_3 + 3x_4 & \leq 100 \\
(10 + d_4)x_1 + 2x_2 + 4x_3 + x_4 & \leq 100 \\
(10 + d_5)x_1 + x_2 + 5x_3 + x_4 & \leq 100 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

where \(d_1, d_2, d_3, d_4, d_5\) are the first 5 digits of your student number.

a) Graph (sketch) the optimal value of the objective function as a function of \(c_1\) for all \(c_1 \in (-\infty, \infty)\). Use the LINDO package or any other software but provide a printout of at least one input and one output. Begin with \(c_1 = 0\) and use ranging to determine an interval for \(c_1\) for which you know the answer. Ranging gives you the interval in which the optimal basis \(B\) is unchanged and the value for \(x_1\) in the optimal solution gives the slope (Why?). Now choose a \(c_1\) outside this interval and repeat. Continue until you know the optimal values for all possible \(c_1\). You might need as many as nine intervals or as few as two intervals depending on your student number.

b) Consider the optimal value of the objective function as a function of the value \(c_1\), say \(f(c_1)\). Show that \(f(c_1)\) is a concave upwards function.

3. (from an old exam) If you are given an optimal primal solution \(x^*\) to an LP and you wish to deduce an optimal dual solution \(y^*\), then you might try to determine \(y^*\) using

(1) Complementary Slackness of \(y^*\) with \(x^*\).

(2) \(y^*\) satisfies constraints (including positivity constraints if any) in dual, i.e. \(y^*\) is a feasible solution of the dual..

Many of our examples in class and quizzes yielded unique optimal \(y^*\) but in general there may be many optimal dual solutions. Are all \(y\) satisfying (1),(2) optimal to the dual? Is every optimal dual solution determined as a solution to (1),(2)? Explain.

4. Let \(A\) be an \(m \times n\) matrix with \(A \geq 0\) and each column of \(A\) has a non-zero positive entry. Let \(b \geq 0\). Then show that the LP

\[
\begin{align*}
\text{max} & \quad c \cdot x \\
Ax & \leq b \\
x & \geq 0
\end{align*}
\]

always has an optimal solution.
a) Show there is an \( x \geq 0 \) with \( Ax < 0 \) if and only if there is an \( x \geq 0 \) with \( Ax \leq -1 \).

Note: we use the definition \((x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)\) if and only if \( x_1 < y_1, x_2 < y_2, \ldots \) \textbf{and} \( x_n < y_n \). This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol > would generally mean \( \geq \) and \( \neq \) but this is not true for matrices or vectors. A vector \( x \) might satisfy \( x \geq 0 \) and also \( x \neq 0 \) and yet still have some 0 entries. Such a vector \( x \) with 0 entries has \( x \neq 0 \).

b) Let \( A \) be an \( m \times n \) matrix. Prove that either:

i) there exists an \( x \geq 0 \) with \( Ax < 0 \) or

ii) there exists \( y \geq 0 \) with \( A^T y \geq 0 \) and \( y \neq 0 \) but not both.

Hint: Extend the idea in a) and use it in setting up a primal dual pair.