1. Give an example of a Linear Program which is infeasible and also its dual is infeasible.

To obtain a primal/dual pair that are both infeasible, we can try making a primal whose dual has the same infeasible constraints. A $2 \times 2$ example suffices.

**Primal:**

$$\begin{align*}
\text{max} & \quad x_1 - 2x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x \geq 0
\end{align*}$$

**Dual:**

$$\begin{align*}
\text{min} & \quad y_1 - 2y_2 \\
\text{subject to} & \quad y_1 - y_2 \leq 1 \\
& \quad -y_1 + y_2 \leq -2 \\
& \quad y \geq 0
\end{align*}$$

2. Since I didn’t write a quiz 2, I will take the corresponding second problem from the practice to quiz 2.

Maximize

$$\begin{align*}
& \quad -2x_1 \\
& \quad -x_1 - x_2 - x_3 \leq 3 \\
& \quad -x_1 - x_3 \leq -2 \\
& \quad -2x_1 - x_2 - x_3 \leq -3
\end{align*}$$

with final dictionary

$$\begin{align*}
x_4 &= 6 \\
x_2 &= 1 \quad -x_5 - x_1 + x_6 \\
x_3 &= 2 \quad +x_5 - x_1 \\
z &= -2 \quad -x_5 - x_1
\end{align*}$$

We read of the final basis as $\{x_4, x_2, x_3\}$.

$$A = \begin{bmatrix}
x_4 \\
x_5 \\
x_6
\end{bmatrix} = \begin{bmatrix}
-1 & -1 & -1 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 \\
-2 & -1 & -1 & 0 & 0 & 1
\end{bmatrix}$$

$$b = \begin{bmatrix}
x_4 \\
x_5 \\
x_6
\end{bmatrix} = \begin{bmatrix}
3 \\
-2 \\
-3
\end{bmatrix}$$

Note that $B^{-1}$ is contained in the final dictionary as the columns of the slack variables $x_4, x_5, x_6$ but beware those pesky minus signs.

Solution:

$$c_N^T - c_B^T B^{-1} A_N = \begin{bmatrix}
x_4 & x_5 & x_6
\end{bmatrix} = \begin{bmatrix}
x_4 & x_5 & x_6
\end{bmatrix}$$
Similarly
\[
x_B = \begin{pmatrix}
  x_4 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
  x_4 & 1 & 0 & -1 \\
x_2 & 0 & 1 & -1 \\
x_3 & 0 & -1 & 0
\end{pmatrix} \begin{pmatrix}
  b \\
x_4 & 3 \\
x_5 & -2 \\
x_6 & -3
\end{pmatrix} \begin{pmatrix}
  x_4 & 1 & 0 & -1 \\
x_2 & 0 & 1 & -1 \\
x_3 & 0 & -1 & 0 \\
x_6 & -2 & 0 & 1
\end{pmatrix}
\]

yielding our final dictionary in a somewhat altered column order.

3. Theorem 5.5 is taken from page 65-66 of V. Chvátal’s book on Linear Programming. Consider the LP:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^m c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m) \\
& \quad x_j \geq 0 \quad (j = 1, \ldots, n)
\end{align*}
\]

**Theorem 5.5.** If (5.24) has at least one non-degenerate basic feasible optimal solution, then there is a positive \( \epsilon \) with the property: If \(|x_i| \leq \epsilon \) for all \( i = 1, \ldots, m \), then the problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^m c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^n a_{ij} x_j \leq b_i + t_i \quad (i = 1, \ldots, m) \\
& \quad x_j \geq 0 \quad (j = 1, \ldots, n)
\end{align*}
\]

has an optimal solution and its optimal value equals \( z^* + \sum_{i=1}^m y_i^* t_i \) with \( z^* \) standing for the optimal value of (5.24) and with \( y_1^*, y_2^*, \ldots, y_m^* \) standing for the optimal solution of its dual.

Now consider the following LP

\[
\begin{align*}
\text{max} & \quad 12x_1 + 20x_2 + 18x_3 \\
& \quad 4x_1 + 6x_2 + 8x_3 \leq 600 \\
& \quad x_1 + (7/2)x_2 + 2x_3 \leq 300 \\
& \quad 2x_1 + 4x_2 + 3x_3 \leq 550
\end{align*}
\]

The final dictionary is:

\[
\begin{align*}
x_1 &= \frac{75}{2} - 2x_3 - (7/16)x_4 + (3/4)x_5 & \text{optimal basis} \\
x_2 &= 75 + (1/8)x_4 - (1/2)x_5 & \quad B^{-1} = x_2 \begin{pmatrix} 7/16 & -3/4 & 0 \\ -1/8 & 1/2 & 0 \end{pmatrix} \\
x_6 &= 175 + x_3 + (3/8)x_4 + (1/2)x_5 & \{x_1, x_2, x_6\} \\
z &= 1950 - 6x_3 - (11/4)x_4 - x_5 & x_6 \begin{pmatrix} -3/8 & -1/2 & 1 \end{pmatrix}
\end{align*}
\]

Theorem 5.5 applies here because the current basic feasible solution is non-degenerate. With \( \Delta b = (t_1, t_2, \ldots, t_3)^T \), the conclusions of Theorem 5.5 are valid if \( B^{-1}(b + \Delta b) \geq 0 \). Thus we need

\[
B^{-1}(b + \Delta b) = \begin{bmatrix} 7/16 & -3/4 & 0 \\ -1/8 & 1/2 & 0 \\ -3/8 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 600 + t_1 \\ 300 + t_2 \\ 550 + t_3 \end{bmatrix} = \begin{bmatrix} 75/2 + (7/16)t_1 - (3/4)t_2 \\ 75 - (1/8)t_1 + (1/2)t_2 \\ 175 - (3/8)t_1 - (1/2)t_2 + t_3 \end{bmatrix} \geq 0.
\]
We must choose \( \epsilon \) so that for all choices for each \( t_i \) satisfying \(-\epsilon \leq t_i \leq \epsilon\), the inequalities are true. From an inequality of the form \( a + a_1 t_1 + a_2 t_2 + a_3 t_3 \geq 0 \) with \( a \geq 0 \), we have form \( a_1 t_1 + a_2 t_2 + a_3 t_3 \geq -a \). Now using \(-\epsilon \leq t_i \leq \epsilon\), we have \( a_1 t_1 + a_2 t_2 + a_3 t_3 \geq -(|a_1|+|a_2|+|a_3|)\epsilon \) which can be achieved by appropriate choices of \( t_1, t_2, t_3 \) (e.g. if \( a_1 < 0 \) take \( t_1 = \epsilon \) and if \( a_1 \geq 0 \) take \( t_1 = -\epsilon \)). Thus \( a_1 t_1 + a_2 t_2 + a_3 t_3 \geq -(|a_1|+|a_2|+|a_3|)\epsilon \geq -a \) and so \( \epsilon \leq \frac{a}{|a_1|+|a_2|+|a_3|} \).

From the first inequality \( \frac{5}{2} + (\frac{7}{16}) t_1 - (\frac{3}{4}) t_2 \geq 0 \) we deduce that the worst case would be to have \( t_1 = -\epsilon \), \( t_2 = \epsilon \) and then deduce that \(-(\frac{7}{16}) + \frac{3}{4})\epsilon \geq \frac{75}{2} \) and hence \( \epsilon \leq \frac{600}{19} \). From the second inequality \( 75 - (\frac{1}{8}) t_1 + (\frac{1}{2}) t_2 \geq 0 \) we deduce that \( \epsilon \leq \frac{600}{3} \). From the third inequality \( 175 - (\frac{3}{8}) t_1 - (\frac{1}{2}) t_2 + t_3 \geq 0 \) we deduce that \( \epsilon \leq \frac{280}{3} \). Thus the largest possible \( \epsilon \) for which if \( |t_i| \leq \epsilon \) for all \( i = 1, 2, 3 \), then the conclusions of Theorem 5.5 hold, is to take \( \epsilon = \frac{600}{19} \).

4. Consider our two phase method in the case that the LP is infeasible. We begin with the primal LP

\[
\begin{align*}
\max \quad & z \\
\text{subject to} \quad & Ax \leq b. \\
\quad & x \geq 0
\end{align*}
\]

We introduce an artificial variable \( x_0 \) and give the new LP (in Phase 1)

\[
\begin{align*}
\max \quad & -x_0 \\
\text{subject to} \quad & [-1 \vert A] \begin{bmatrix} x_0 \\ x \end{bmatrix} \leq b, \\
\quad & x \geq 0, x_0 \geq 0
\end{align*}
\]

a) Its dual is

\[
\begin{align*}
\min \quad & b \cdot y \\
\text{subject to} \quad & -(1)^T y \geq -1, \\
\quad & A^T y \geq 0, \\
\quad & y \geq 0
\end{align*}
\]

We have assumed the maximum value of the objective function in the new LP is strictly negative and so by Strong Duality, the objective function in the dual is strictly negative. Hence for an optimal dual solution \( y^* \) we have \( b \cdot y^* < 0 \) and \( y^* \) feasible and hence \( A^T y^* \geq 0 \) and \( y^* \geq 0 \). These conditions yield that if we take the sum of \( y^*_i \) times the \( i \)th inequality of the primal for \( i = 1, 2, \ldots, m \), then we obtain a new inequality (the \( y^*_i \geq 0 \) ensure no inequalities flip) whose coefficients for each \( x_j \) are all positive (because \( A^T y^* \geq 0 \)) and whose right hand side is strictly negative (\( b \cdot y^* < 0 \)) which shows that the inequalities have no feasible solution.

b) An optimal primal solution for our new LP has \( x_0 > 0 \) and hence by Complementary Slackness we deduce that for any optimal dual solution \( y^* \) we have that the inequality \( -(1)^T y \geq -1 \) is an equality \( -(1)^T y^* = -1 \) which for \( y^* = (y^*_1, y^*_2, \ldots, y^*_m)^T \) becomes \( y^*_1 + y^*_2 + \cdots + y^*_m = 1 \).

5.

a) Let \( A \) be an \( m \times n \) matrix and let \( u \) be an \( n \times 1 \) vector (of upper bounds). We wish to show that either:

i) there exists an \( x \) with \( Ax \leq b, x \leq u \)

or
ii) there exists vectors \( y, z \) with \( A^T y + z = 0, \ y, z \geq 0, \ b \cdot y + u \cdot z < 0 \) but not both.

**Proof:** Consider the following primal dual pair of Linear Programs:

\[
\begin{align*}
\text{max} & \quad 0 \cdot x \\
\text{primal:} & \quad A x \leq b, \\
& \quad x \leq u
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad b \cdot y + u \cdot z \\
\text{dual:} & \quad A^T y + I z = 0, \\
& \quad y \geq 0, \ z \geq 0
\end{align*}
\]

We note that \( y = 0 \) and \( z = 0 \) yields a feasible solution to the Dual (these vectors have \( m \) and \( n \) coordinates respectively). By the Fundamental Theorem of Linear Programming, we deduce that the dual either has an optimal solution or is unbounded.

**CASE 1:** Dual has an optimal solution \( y^*, z^* \).

Thus, by Strong Duality, the primal has an optimal solution \( x^* \) which is a feasible solution and hence i) holds. Also \( 0 = 0 \cdot x^* = b \cdot y^* + u \cdot z^* \) which means by Weak Duality that every feasible solution \( y, z \) to the dual has \( 0 \leq b \cdot y + u \cdot z \) and hence ii) does not hold.

**CASE 2:** Dual is unbounded.

Thus there is a feasible solution \( y, z \) to the dual has \( b \cdot y + u \cdot z \leq -1 < 0 \) and hence ii) holds. The dual being unbounded implies, using Weak Duality, that the primal has no feasible solution and hence i) does not hold.

Thus we have established the theorem in both cases.

b) Use a) to show that the system of equalities with bounded variables given below has no solution:

\[
\begin{align*}
-x_1 + x_2 & \leq 2 \\
x_1 - x_2 & \leq -2 \\
-x_1 - 3x_2 & \leq -18 \\
x_1 & \leq 2 \\
x_2 & \leq 2
\end{align*}
\]

We can guess and take \( y_1 = 1, \ y_2 = 1, \ y_3 = 1, \ z_1 = 1, \) and \( z_2 = 3 \). We have

\[
\begin{bmatrix}
-1 & 1 & -1 & 1 & 0 \\
1 & -1 & -3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

and \( 2y_1 - 2y_2 - 18y_3 + 2z_1 + 2z_2 = -10 < 0 \) and so by c), no solution to the 5 inequalities in \( x_1, x_2 \) exist. The choices for \( y, z \) are not unique. A better solution is to take \( y_1 = 0, \ y_2 = 0, \ y_3 = 1, \ z_1 = 1, \) and \( z_2 = 3 \) where it is still true \( 2y_1 - 2y_2 - 18y_3 + 2z_1 + 2z_2 = -10 < 0 \). This reveals that the 3 inequalities \( -x_1 - 3x_3 \leq -18, \ x_1 \leq 2, \ x_2 \leq 2 \) are infeasible.