The three inequalities
\[ x - y \leq -2 \quad -x + 2y \leq 5 \quad -x - y \leq -6 \]
have no solution \( x, y \) with \( x, y \geq 0 \):
We apply Phase One:

\[
\begin{align*}
x_1 &= -2 -x +y +x_0 \\
x_2 &= 5 +x -2y +x_0 \\
x_3 &= -6 +x +y +x_0 \\
w &= -x_0
\end{align*}
\]

Special pivot to feasibility, \( x_0 \) enters and \( x_3 \) leaves.

\[
\begin{align*}
x_1 &= 4 -2x +x_3 \\
x_2 &= 11 -3y +x_3 \\
x_0 &= 6 -x -y +x_3 \\
w &= -6 +x +y -x_3
\end{align*}
\]

(I used Anstee’s rule with \( x \) before \( y \)) \( x \) enters and \( x_1 \) leaves.

\[
\begin{align*}
x &= 2 -1/2x_1 +1/2x_3 \\
x_2 &= 11 -3y +x_3 \\
x_0 &= 4 +1/2x_1 -y +1/2x_3 \\
w &= -4 -1/2x_1 +y -1/2x_3
\end{align*}
\]

\( y \) enters and \( x_2 \) leaves.

\[
\begin{align*}
x &= 2 -1/2x_1 +1/2x_3 \\
y &= 11/3 -1/3x_2 +1/3x_3 \\
x_0 &= 1/3 +1/2x_1 +1/3x_2 +1/6x_3 \\
w &= -1/3 -1/2x_1 -1/3x_2 -1/6x_3
\end{align*}
\]

We are at optimality with \( w = -1/3 \). Thus we cannot drive \( x_0 \) to 0. Using the magic coefficients, the negatives of the coefficients of the slack variables \( x_1, x_2, x_3 \), namely \( \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \) we apply these to the three inequalities:

\[
\frac{1}{2}(x - y \leq -2) + \frac{1}{3}(-x + 2y \leq 5) + \frac{1}{6}(-x - y \leq -6)
\]

to get the contradiction:

\[
0 \leq \frac{-2}{6}.
\]

Thus the truth of three inequalities (without even using the positivity of \( x, y \)) yields an impossible situation so there can be no solution \( x, y \) to the three inequalities. You will note that all three inequalities were involved on the final answer. That is because any two of the inequalities will yield a feasible solution.
2.

a) First we transform LP3 into standard inequality form and call it LP7. We must replace the equality by two inequalities and substitute \( x_2 = x'_2 - x''_2 \) where \( x'_2, x''_2 \geq 0 \).

\[
LP7 \quad \text{max} \quad 2x_1 + 4x'_2 - 4x''_2 \\
3x_1 - 7x'_2 + 7x''_2 \leq 6 \\
-3x_1 + 7x'_2 - 7x''_2 \leq -6 \\
6x_1 + 2x'_2 - 2x''_2 \leq 5
\]

The dual of an LP7 is:

\[
LP8 \quad \text{min} \quad 6z_1 - 6xz_2 + 5z_3 \\
3z_1 - 3z_2 + 6x_3 \geq 2 \\
-7z_1 + 7z_2 + 2z_3 \geq 4 \\
7z_1 - 7z_2 - 2z_3 \geq -4
\]

Now we can see that LP8 equivalent to LP4 by combining the last two inequalities into the equivalent equality \(-7z_1 + 7z_2 + 2z_3 = 4\) and using the variable substitutions \( z_1 - z_2 = y_1 \) and \( z_3 = y_2 \) so that \( y_1 \) is unconstrained and \( y_2 \geq 0 \).

In general our transformations may have affected the value of the objective function when we replace \( z \) by \(-z\) (going from a min to a max) or when we delete a constant. But this does not occur in this problem.

b) We proceed as above transforming LP5 into LP9

\[
LP9 \quad \text{max} \quad a \cdot x + b \cdot y' - b \cdot y'' \\
A\cdot x + B\cdot y' - B\cdot y'' \leq c \\
-A\cdot x - B\cdot y' + B\cdot y'' \leq -c \\
C\cdot x + D\cdot y' - D\cdot y'' \leq d
\]

The dual LP10 is

\[
LP10 \quad \text{min} \quad c \cdot z' - c \cdot z'' + d \cdot t \\
A^T\cdot z' - A^T\cdot z'' + C^T\cdot t \geq a \\
B^T\cdot z' - B^T\cdot z'' + D^T\cdot t \geq b \\
-B^T\cdot z' + B^T\cdot z'' - D^T\cdot t \geq -b
\]

We can see that LP10 can be obtained from LP6 by realizing that the inequalities \( B^Tz' - B^Tz'' + D^Tt \geq b \) and \(-B^Tz' + B^Tz'' - D^Tt \geq -b \) yields the equalities \( B^Tz' - B^Tz'' + D^Tt = b \) and then we replace \( z' - z'' = z \) and have that each variable in \( z \) is free.

3. We say a set \( C \) of points in \( \mathbb{R}^n \) is convex if for every pair \( x, y \in C \), all points on the line segment joining \( x \) and \( y \) are in \( C \). Thus \( C \) is a convex set if for every pair \( x, y \in C \) and any \( \lambda \in (0, 1) \) we have \( \lambda x + (1 - \lambda)y \in C \). Note that \( \lambda x + (1 - \lambda)y = y + \lambda(x - y) \) which for \( \lambda \in [0, 1] \), yields the line segment joining \( x \) and \( y \). Let \( A \) be an \( m \times n \) matrix and \( b \) a given vector in \( \mathbb{R}^m \). Show that

\[
F = \{ x \in \mathbb{R}^n : Ax \leq b, \quad x \geq 0 \}
\]

is a convex set.
We check that for \( \lambda \in [0, 1] \), that \( \lambda \geq 0 \) and \( (1-\lambda) \geq 0 \). Let \( x, y \in F \). Then we have \( Ax \leq b \), \( x \geq 0 \), \( Ay \leq b \) and \( y \geq 0 \).

We note that if \( u, v \) are vectors and \( c \geq 0 \) a scalar, then if \( u \geq v \), we have \( cu \geq cv \). You can check entry by entry to see this. Now \( x \geq 0 \) implies \( \lambda x \geq \lambda 0 = 0 \) and \( y \geq 0 \) implies \( (1-\lambda)y \geq (1-\lambda)0 = 0 \). Then \( \lambda x + (1-\lambda)y \geq 0 \). Also \( Ax \leq b \) implies \( A(\lambda x) = \lambda(Ax) \leq \lambda b \) and \( Ay \leq b \) implies \( A((1-\lambda)y) = (1-\lambda)(Ay) \leq (1-\lambda)b \). Then we obtain

\[
A(\lambda x + (1-\lambda)y) = A(\lambda x) + A((1-\lambda)y) \leq \lambda b + (1-\lambda)b = b
\]

We now conclude \( (\lambda x + (1-\lambda)y) \in F \) and so \( F \) is convex.

4. Consider an LP: \( \max c \cdot x \) such that \( Ax \leq b \) and \( x \geq 0 \). Assume the LP has two feasible solutions \( u \) and \( v \) with \( c \cdot u = 10 \) and \( c \cdot v = 50 \). Show that there exists a feasible solution \( w \) to the LP with \( c \cdot w = 20 \). (Note that \( \lambda a + (1-\lambda)b \) for \( \lambda \in [0, 1] \) is a weighted average of \( a \) and \( b \).)

We note that \( 20 = \frac{3}{4} \cdot 10 + \frac{1}{4} \cdot 50 \) where \( 1 - \frac{3}{4} = \frac{1}{4} \). Let \( w = \frac{3}{4}u + \frac{1}{4}v \). By our previous question \( w \in F \). Now we again use linearity and note that

\[
c \cdot w = c \cdot \left( \frac{3}{4}u + \frac{1}{4}v \right) = \frac{3}{4}c \cdot u + \frac{1}{4}c \cdot v
\]

\[
= \frac{3}{4} \cdot 10 + \frac{1}{4} \cdot 50 = 20.
\]

5. Consider an LP: \( \max c \cdot x \) such that \( Ax \leq b \) and \( x \geq 0 \). Assume the LP is unbounded, in fact assume there is a parametrized set of feasible solutions \( w(t) = y + tu \) for some choice of \( y, u \) where

\[
\lim_{t \to \infty} c \cdot w(t) = \infty
\]

(this is what our simplex algorithm spits out.) Show that this means \( u \geq 0 \), \( Au \leq 0 \) and \( c \cdot u > 0 \).

To show that \( u \geq 0 \) assume that it is not true that \( u \geq 0 \) and hence assume that there is some index \( j \) such that the \( j \)th entry of \( u \) is strictly negative, namely \( u_j < 0 \). But if we let \( y_j \) denote the \( j \)th entry of \( y \) then the \( j \)th entry of \( w(t) \) is \( y_j + tu_j \). We note that \( y_j + tu_j < 0 \) for \( t > -y_j/u_j \). This is a contradiction to the assertion \( w(t) \geq 0 \) for all \( t \geq 0 \). We conclude \( u \geq 0 \).

A similar argument shows that \( Au \leq 0 \). In particular if there is some index \( j \) such that the \( j \)th entry of \( Au \) is strictly positive, say \( d > 0 \), and we let \( a \) be the \( j \)th entry of \( Ay \), then the \( j \)th entry of \( Aw(t) \) is \( a + td \) which is larger than the \( j \) entry of \( b \) for suitably large \( t \). This contradicts that \( Aw(t) \leq b \) for all \( t \geq 0 \) and so establishes that \( Au \leq 0 \).

We use linearity to compute that \( c \cdot w(t) = c \cdot y + tc \cdot u \). We are given that \( \lim_{t \to \infty} c \cdot w(t) = \infty \) and yet \( \lim_{t \to \infty} c \cdot y = c \cdot y \) so we have that \( \lim_{t \to \infty} tc \cdot u = \infty \). This yields that \( c \cdot u > 0 \).

Also show the converse, namely that if an LP \( \{ \max c \cdot x \text{ such that } Ax \leq b \text{ and } x \geq 0 \} \) has a feasible solution \( u \) and there exists a vector \( v \) with \( v \geq 0 \), \( Av \leq 0 \) and \( c \cdot v > 0 \), then the LP is unbounded.

Our hypothesis also ensures \( Au \leq b \) and \( u \geq 0 \). We now set \( w(t) = u + tv \). We verify that \( w(t) \) is feasible for \( t \geq 0 \).

\[
A(w(t)) = A(u + tv) = Au + tAv \leq b + 0 = b
\]
since $A u \leq b$ and $A v \leq 0$ and so $tA v \leq 0$ for $t \geq 0$. Similarly $w(t) = u + tv \geq 0$ since $u \geq 0$ and $v \geq 0$ so $tv \geq 0$ for $t \geq 0$. Finally

$$\lim_{t \to \infty} c \cdot w(t) = \lim_{t \to \infty} c \cdot u + t(c \cdot v) = c \cdot u + c \cdot v \lim_{t \to \infty} t = \infty$$

using $c \cdot v > 0$. 