Math 340 Midterm Solutions Wednesday, February 24, 2016

Explain your work. Name LP theorems as you use them.

1. [30pts] (there were two versions) Solve the following LP using our two phase method with Anstee’s rule. You will need a fake pivot to feasibility and two more pivots in Phase One. In Phase Two you will need one pivot.

Maximize \[-2x_1 - x_2 - x_3\]
\[-x_1 - x_3 \leq -1 \quad x_1, x_2, x_3 \geq 0\]
\[-x_1 - x_2 \leq -2\]
\[-x_2 + x_3 \leq -1\]

Give two optimal solutions (they will both have the same objective function value).

Phase One:

\[x_4 = -1 + x_1 + x_3 + x_0\]
\[x_5 = -2 + x_1 + x_2 + x_0\]
\[x_6 = -1 + x_2 - x_3 + x_0\]
\[w = -x_0\]

\[x_0 \text{ enters and } x_5 \text{ leaves (Non-standard pivot to feasibility)}\]

\[x_4 = 1 - x_2 + x_3 + x_5\]
\[x_0 = 2 - x_1 - x_2 + x_5\]
\[x_6 = 1 - x_1 - x_3 + x_5\]
\[w = -2 + x_1 + x_2 - x_5\]

\[x_1 \text{ enters and } x_6 \text{ leaves}\]

\[x_4 = 1 - x_2 + x_3 + x_5\]
\[x_0 = 1 + x_6 - x_2 + x_3\]
\[x_1 = 1 - x_6 - x_3 + x_5\]
\[w = -1 - x_6 + x_2 - x_3\]

\[x_2 \text{ enters and } x_0 \text{ leaves}\]

\[x_4 = 0 - x_6 + x_0 + x_5\]
\[x_2 = 1 + x_6 - x_0 + x_3\]
\[x_1 = 1 - x_6 - x_3 + x_5\]
\[w = 0 - x_0\]

End of Phase One. Delete \(x_0, w\) and introduce \(z\). Note \(z = -2x_1 - x_2 - x_3 = -2(1 - x_6 - x_3 + x_5) - (1 + x_6 + x_3) - x_3\).

\[x_4 = 0 - x_6 + x_5\]
\[x_2 = 1 + x_6 + x_3\]
\[x_1 = 1 - x_6 - x_3 + x_5\]
\[z = -3 + x_6 - 2x_5\]

\[x_6 \text{ enters and } x_4 \text{ leaves (a degenerate pivot)}\]

\[x_6 = 0 - x_4 + x_5\]
\[x_2 = 1 - x_4 + x_3 + x_5\]
\[x_1 = 1 + x_4 - x_3\]
\[z = -3 - x_4 - x_5\]
We now have an optimal solution \((1,1,0,0,0,0)\) with \(z = -3\). We can obtain an additional solution by increasing \(x_3\) by 1 to \(x_3 = 1\) which yields an optimal solution \((0,2,1,0,0,0)\) with \(z = -3\).

**Version 2:**

\[
\begin{align*}
\text{Maximize} & \quad 2x_1 - 2x_2 + x_3 \\
\end{align*}
\]

\[
\begin{align*}
2x_1 - x_2 & \leq -3 & x_1, x_2, x_3 \geq 0 \\
-x_2 + x_3 & \leq -2 \\
-2x_2 - x_3 & \leq -5
\end{align*}
\]

Give the optimal solution and one additional optimal solution.

**Phase One:**

\[
\begin{align*}
x_4 & = -3 & -2x_1 & + x_2 & + x_0 \\
x_5 & = -2 & -x_1 & + x_2 & - x_3 & + x_0 \\
x_6 & = -5 & +2x_2 & + x_3 & + x_0 \\
w & = & & & -x_0
\end{align*}
\]

\(x_0\) enters and \(x_6\) leaves (**Non-standard pivot to feasibility**)

\[
\begin{align*}
x_4 & = 2 & -2x_1 & - x_2 & - x_3 & + x_6 \\
x_5 & = 3 & -x_1 & - x_2 & -2x_3 & + x_6 \\
x_0 & = 5 & -2x_2 & - x_3 & + x_6 \\
w & = -5 & +2x_2 & + x_3 & - x_6
\end{align*}
\]

\(x_2\) enters and \(x_4\) leaves

\[
\begin{align*}
x_2 & = 2 & -2x_1 & - x_4 & - x_3 & + x_6 \\
x_5 & = 1 & +x_1 & + x_4 & - x_3 \\
x_0 & = 1 & +4x_1 & +2x_4 & + x_3 & - x_6 \\
w & = -1 & -4x_1 & -2x_4 & - x_3 & + x_6
\end{align*}
\]

\(x_6\) enters and \(x_0\) leaves

\[
\begin{align*}
x_2 & = 3 & +2x_1 & + x_4 & - x_0 \\
x_5 & = 1 & +x_1 & + x_4 & - x_3 \\
x_6 & = 1 & +4x_1 & +2x_4 & + x_3 & - x_0 \\
w & = 0 & & & & -x_0
\end{align*}
\]

End of Phase One. Delete \(x_0, w\) and introduce \(z\). Note \(z = 3x_1 - 2x_2 + x_3 = 3x_1 - 2(3 + 2x_1 + x_4) + x_3 = -6 - x_1 - 2x_4 + x_3\).

\[
\begin{align*}
x_2 & = 3 & +2x_1 & + x_4 \\
x_5 & = 1 & +x_1 & + x_4 & - x_3 \\
x_6 & = 1 & +4x_1 & +2x_4 & + x_3 \\
z & = -6 & - x_1 & -2x_4 & + x_3
\end{align*}
\]

\(x_3\) enters and \(x_5\) leaves

\[
\begin{align*}
x_2 & = 3 & +2x_1 & + x_4 \\
x_3 & = 1 & +x_1 & + x_4 & - x_5 \\
x_6 & = 2 & +5x_1 & +3x_4 & - x_5 \\
z & = -5 & - x_4 & - x_5
\end{align*}
\]

We now have an optimal solution \((0,3,1,0,0,2)\) with \(z = -5\).
We can obtain an additional solution by increasing $x_1$ say $x_1 = 1$ which yields an optimal solution $(1, 5, 2, 0, 0, 7)$ with $z = -5$.

2.[20pts] Consider the following LP

$$\begin{align*}
\text{max} & \quad 3x_1 + 5x_2 + 2x_3 \\
\text{subject to} & \quad x_1 + x_2 + 2x_3 \leq 5, \quad x_1, x_2, x_3 \geq 0 \\
& \quad 2x_1 + x_3 \leq 4 \\
& \quad x_1 + 2x_2 + 4x_3 \leq 9
\end{align*}$$

We are given that an optimal dual solution is $y_1 = 1/2$, $y_2 = 0$, $y_3 = 5/2$. The dual LP is

$$\begin{align*}
\text{Minimize} & \quad 5y_1 + 4y_2 + 9y_3 \\
\text{subject to} & \quad y_1 + 2y_2 + y_3 \geq 3, \quad y_1, y_2, y_3 \geq 0 \\
& \quad y_1 + 2y_2 \geq 5 \\
& \quad 2y_1 + y_2 + 4y_3 \geq 11
\end{align*}$$

We are given that $y_1^* = 1/2$, $y_2^* = 0$, $y_3^* = 5/2$ is an optimal solution to the dual of this LP. Let $x_1^*, x_2^*, x_3^*$ denote an optimal primal solution.

- $y_1^* = 1 > 0$ implies, by Complementary Slackness, that $x_1^* + x_2^* + 2x_3^* = 5$
- $y_2^* = 2 > 0$ implies, by Complementary Slackness, that $x_1^* + 2x_2^* + 4x_3^* = 9$
- $y_1^* + 2y_3^* = 11/2 > 5$ implies by Complementary Slackness that $x_2^* = 0$.

We solve to obtain $x_1^* = 1, x_2^* = 0, x_3^* = 2$.

Since the values for $x_1^*, x_2^*, x_3^*$ were determined by the three equations (in $x_1^*, x_2^*, x_3^*$) determined by Complementary Slackness and the resulting system of equations had a unique solution so the optimal solution to the primal must be the unique solution.

3.[20 points] (Version 1) We are given $A, b, c$, current basis and $B^{-1}$. Determine, using our revised simplex methods, the next entering variable (there is one!) and the next leaving variable (if there is one). If there is no leaving variable, give a parametric solution that shows that the LP is unbounded. If there is a leaving variable give the next basic feasible solution and the new matrix $B$ (don’t compute new $B^{-1}$).

$$\begin{align*}
A &= \begin{pmatrix}
1 & 2 & 1 & 1 & 1 & 0 & 0 \\
-1 & 3 & 1 & 2 & 0 & 1 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 1
\end{pmatrix} \\
\text{basis: } & \begin{pmatrix}
x_5 \\
x_6 \\
x_7
\end{pmatrix} \begin{pmatrix}
4 \\
6 \\
-1
\end{pmatrix} \\
B^{-1} &= \begin{pmatrix}
x_7 \\
x_3 \\
x_4
\end{pmatrix} \begin{pmatrix}
-1 & 1 & 1 \\
2 & -1 & 0 \\
-1 & 1 & 0
\end{pmatrix}
\end{align*}$$

$$c_N - c_B B^{-1} A_N = \begin{pmatrix}
x_1 & x_2 & x_5 & x_6 \\
0 & 3 & 4 & 0
\end{pmatrix} - \begin{pmatrix}
x_1 & x_2 & x_5 & x_6 \\
0 & 3 & 4 & 0
\end{pmatrix} = \begin{pmatrix}
x_1 & x_2 & x_5 & x_6 \\
0 & 3 & -2 & -1
\end{pmatrix}$$
Thus we choose \( x_2 \) to enter.

\[
B^{-1}\mathbf{b} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad B^{-1}\mathbf{A}_2 = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.
\]

Now \( B^{-1}\mathbf{b} - B^{-1}\mathbf{A}_2x_2 \geq 0 \) implies \( x_2 \leq 2 \) and \( x_3 \) leaves the basis (tied with \( x_4 \) but we use Anstee’s rule). The new solution has ‘new’ \( \mathbf{B} \) as follows:

\[
\begin{pmatrix} x_7 & x_2 & x_4 \\ x_5 & 0 & 2 & 1 \\ x_6 & 0 & 3 & 2 \\ x_7 & 1 & -1 & -1 \end{pmatrix}
\]

and the new solution has \( x_2 = 2, x_5 = 1, x_3 = 0, x_4 = 0 \) and \( x_1 = x_5 = x_6 = 0 \). It is a degenerate basic feasible solution but the question did not ask you this.

(Version 2) We are given \( \mathbf{A}, \mathbf{b}, \mathbf{c} \), current basis and \( B^{-1} \). Determine, using our revised simplex methods, the next entering variable (there is one!) and the next leaving variable (if there is one). If there is no leaving variable, give a parametric solution that shows that the LP is unbounded. If there is a leaving variable give the next basic feasible solution and the new matrix \( \mathbf{B} \) (don’t compute new \( B^{-1} \)).

We proceed as usual:

\[
\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 2 & 4 & 6 & 1 & 1 & 0 \\ -1 & 1 & 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \end{pmatrix}
\]

\[
\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}
\]

\[
\mathbf{b}
\]

\[
\begin{pmatrix} \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \end{pmatrix}
\]

\[
\begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & -4 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}
\]

\[
\mathbf{c}^T
\]

\[
\begin{pmatrix} -2 & 1 & 3 & 1 & 0 & 0 & 0 \end{pmatrix}
\]

Thus we choose \( x_3 \) to enter.

\[
B^{-1}\mathbf{b} = \begin{pmatrix} 1 & -4 & -2 \\ 0 & -1 & -2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad B^{-1}\mathbf{A}_3 = \begin{pmatrix} 1 & -4 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.
\]

Now \( B^{-1}\mathbf{b} - B^{-1}\mathbf{A}_3x_3 \geq 0 \) only for \( x_3 = 0 \), namely \( x_3 \) can increase from 0 to 0 in a degenerate pivot with \( x_4 \) leaving the basis. Thus the new basic feasible solution is the same with \( x_5 = 2, x_2 = 1 \) and the rest zero. The new basis yields

\[
\begin{pmatrix} x_5 & x_2 & x_3 \\ x_5 & 1 & 2 & 4 \\ x_6 & 0 & 1 & 0 \\ x_7 & 0 & -1 & 1 \end{pmatrix}
\]
4.[10 pts]
a) State the Theorem of Complementary Slackness. Let \( x \) be a feasible solution to the primal and let \( y \) be a feasible solution to the dual. Then \( x \) is optimal to primal and \( y \) is optimal to dual if and only if the conditions of complementary slackness hold:

\[
x_j \cdot j\text{th dual slack} = 0 \text{ for } j = 1, 2, \ldots, n
\]
and

\[
y_i \cdot i\text{th primal slack} = 0 \text{ for } i = 1, 2, \ldots, m
\]
b) (version 1) Explain how we compute \( c^T B B^{-1} \) using gaussian elimination (and not by computing \( B^{-1} \)).

The idea here is to solve for \( y^T \) in the equation \( c^T B B^{-1} = y^T \) which we first multiply on the right by \( B \) to obtain \( c^T_B = y^T B \) which, after taking transposes becomes the more familiar equation \( B^T y = c_B^T \). Then use typical gaussian elimination applied to \( B^T \) and \( c_B \) to solve for \( y \).

b) (version 2) What is cycling in the context of Linear Programming and why do we study it?

Cycling is a sequence of pivots that begins at one basis \( B \) and returns to that basis after some number of pivots. Assuming you are following some standard pivot rules, this means you will enter an infinite loop of pivots and the simplex algorithm will never terminate. We need for practical and theoretical reasons to have the algorithm terminate and not get stuck in an infinite loop.

5.[20pts] Let \( A \) be an \( m \times n \) matrix and let \( b \) be an \( n \times 1 \) vector. Assume that for all \( y \) for which \( y \geq 0 \) and \( A y = 0 \), we have \( b \cdot y \geq 0 \). Prove that there exists an \( x \) satisfying \( A^T x \leq b \).

Consider the following primal/dual pair of LP’s

<table>
<thead>
<tr>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max \ 0 \cdot x )</td>
<td>( \min \ b \cdot y )</td>
</tr>
<tr>
<td>( A^T x \leq b )</td>
<td>( A y = 0 )</td>
</tr>
<tr>
<td>( x ) free</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

Our hypothesis that for all \( y \) with \( A y = 0 \), we have \( b \cdot y \geq 0 \) can now be interpreted as saying that for our dual, every feasible solution \( y \) has its objective function positive and hence bounded from below. We deduce that the dual is not unbounded. We also note that dual is not infeasible since \( y = 0 \) is a feasible solution. Hence by the fundamental theorem of linear programming, we know that the dual has an optimal solution \( y^* \). But now by Strong Duality, we know the primal has an optimal solution \( x^* \) and hence \( x^* \) is feasible and hence \( A^T x^* \leq b \). This is what we were asked to prove.