1. Consider an LP which has as its first constraint

\[-x_1 + x_2 - x_3 - x_4 \leq 1.\]

Assume that you know (based on other constraints) that \(x_2 = 0\). Show that the dual variable \(y_1\) associated with the first constraint is zero.

We immediately deduce that \(-x_1 + x_2 - x_3 - x_4 = -x_1 - x_3 - x_4\). Now making the assumption that all variables are positive (true for LP’s in standard inequality form) we have \(-x_1 - x_3 - x_4 \leq 0 < 1\) and hence the slack variable of this constraint is at least 1. Now, by the Theorem of Complementary Slackness, we deduce that \(y_1 = 0\).

2. I used \(d_1d_2d_3d_4d_5 = 54321\) and obtained

\[
\begin{align*}
\text{max} & \quad c_1x_1 + 7x_2 + 7x_3 + 11.1x_4 \\
15x_1 + 4.5x_2 + 1.2x_3 + 9x_4 & \leq 100 \\
14x_1 + 4x_2 + 1x_3 + 8.2x_4 & \leq 100 \\
13x_1 + 3x_2 + 3x_3 + 3x_4 & \leq 100 \\
12x_1 + 2x_2 + 4x_3 + x_4 & \leq 100 \\
11x_1 + x_2 + 5x_3 + x_4 & \leq 100
\end{align*}
\]

\(x_1, x_2, x_3, x_4 \geq 0\)

There were three intervals:

\(c_1 \in (-\infty, 31.244]\) gives solution \((0, 15.55, 16.66, 1.11)\) with \(z = 237.88\)

\(c_1 \in [31.244, 87.5]\) gives solution \((6.149, 0, 6.472, 0)\) with \(z = 6.149c_1 + 45.304\)

\(c_1 \in [87.5, +\infty)\) gives solution \((6.666, 0, 0, 0)\) with \(z = 6.666c_1\)

You may wish to note that since we are only changing \(c_1\) and we look at intervals in which the optimal basis is unchanged, then the optimal solution \(B^{-1}b\) stays unchanged in that interval. Thus the slope of the line in an interval is the value of \(x_1\). The extent of the interval is found by checking the objective function ranging for the coefficient of \(x_1\) \((c_1)\) which gives the interval (around the current value of \(c_1\)) for which the current basis remains an optimal basis and so for which the solution remains unchanged.

One lucky (?) student had a degeneracy arise \((d_1d_2d_3d_4d_5 = 41441)\) and so there were two different bases and two associated intervals for which the value of \(x_1\) and the other variables were the same in both and hence in the graph, the intervals could be combined into one interval.

b) To show that the (piecewise linear) curve is concave upwards it suffices to show that for each interval where we have computed \(z = x_1c_1 + \text{constant}\) for \(a \leq c_1 \leq b\) then \(z \geq x_1c_1 + \text{constant}\) for all values of \(c_1\). This is easily seen to be true since the feasible solution which achieves \(z = x_1c_1 + \text{constant}\) is a feasible solution regardless of \(c_1\) and so we have a feasible solution to our LP of value \(x_1c_1 + \text{constant}\) for all values of \(c_1\) and hence the optimal value of the objective function is at least this big.

3. (from an old exam) If you are given an optimal primal solution \(x^*\) to an LP and you wish to deduce an optimal dual solution \(y^*\), then you might try to determine \(y^*\) using

1. Complementary Slackness of \(y^*\) with \(x^*\).
(2) $y^*$ satisfies constraints (including positivity constraints if any) in dual, i.e. $y^*$ is a feasible solution of the dual.

All $y^*$ satisfying (1),(2) are feasible to the dual by (2) and then since $x^*$ is feasible and (2) holds (which is complementary slackness) and so by the Complementary Slackness Theorem, we deduce $y^*$ is optimal to the dual (and $x^*$ is optimal to the primal).

Every optimal dual solution $y^*$ is feasible and hence satisfies (2) and also by the other direction of the Complementary Slackness Theorem, since $x^*$ and $y^*$ are optimal to their respective LP’s, we have that (1) holds.

This question is in essence asking for a restatement of the Complementary Slackness Theorem.

4. Let $A$ be an $m \times n$ matrix with $A \geq 0$ and each column of $A$ has a non-zero positive entry. Let $b \geq 0$. Then show that the LP
\[
\begin{align*}
\max c \cdot x \\
Ax &\leq b \\
x &\geq 0
\end{align*}
\]
always has an optimal solution.

The conditions on $A,b$ are frequently sensible for constraints that are based on available resources. In most practical problems it is possible to show that unboundedness cannot happen.

First we note that $x = 0$ shows that the LP has a feasible solutions using that $b \geq 0$.

Second we show that the LP is bounded. Let $a_{ij}$ denote the entry in $A$ in row $i$ and column $j$. Then the $i$th constraint becomes
\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i
\]
I wish to show that the possible values for $x_\ell$ are bounded. Assume that in constraint $k$, that $a_{k\ell} > 0$ (some such $k$ exists by hypothesis). Then
\[
\sum_{j=1}^{n} a_{kj}x_j \leq b_k \text{ and so } a_{k\ell}x_\ell \leq b_k - \sum_{j=1, j \neq \ell}^{n} a_{kj}x_j
\]
Given that $a_{ij} \geq 0$ and $x_j \geq 0$, we deduce that
\[
0 \leq x_\ell \leq \frac{b_k}{a_{k\ell}}
\]
For $c_\ell \geq 0$, we have $0 \leq c_\ell x_\ell \leq c_\ell \cdot \frac{b_k}{a_{k\ell}}$

and for $c_\ell < 0$, we have $c_\ell \cdot \frac{b_k}{a_{k\ell}} \leq c_\ell x_\ell \leq 0$.

Thus there exists lower and upper bounds $g_\ell,f_\ell$ with $g_\ell \leq c_\ell \cdot x_\ell \leq f_\ell$. we deduce that for every feasible $x$ to the LP,
\[
\sum_{j=1}^{n} g_j \leq c \cdot x \leq \sum_{j=1}^{n} f_j
\]
Now given the LP is bounded and has a feasible solution, we deduce by the Fundamental Theorem of LP that the LP has an optimal solution.
An alternate strategy was to consider the dual and show that it has an optimal solution. The dual is

$$\min b \cdot y$$

$$A^T y \geq c$$

$$y \geq 0$$

Given that $b \geq 0$ and $y \geq 0$ we have $b \cdot y \geq 0$. So the dual is bounded. But does it have a feasible solution? In the matrix $A^T$ we know that every row has a nonzero entry. Let the $j$th constraint of the dual be

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j$$

This inequality is always true for $c_j \leq 0$ so assume $c_j > 0$. Then assume $a_{kj} > 0$. We choose $y_j$ to satisfy $y_j \geq c_j/a_{kj}$ and again we deduce the inequality is satisfied. We then choose $y$ ‘large enough’ for each $y_j$ to satisfy these constraints (there will be $n$ conditions from the $n$ inequalities and also the $m$ inequalities $y \geq 0$). Thus the dual is feasible and so, by the Fundamental Theorem of LP’s, the dual has an optimal solution and then, by Strong Duality, the original LP (the primal) has an optimal solution.

5.

a) Show there is an $x \geq 0$ with $Ax < 0$ if and only if there is an $x \geq 0$ with $Ax \leq -1$.

Note: we use the definition $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_1 < y_1, x_2 < y_2, \ldots$ and $x_n < y_n$. This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol $>$ would generally mean $\geq$ and $\neq$ but this is not true for matrices or vectors. A vector $x$ might satisfy $x \geq 0$ and also $x \neq 0$ and yet still have some 0 entries. Such a vector $x$ with 0 entries has $x \not> 0$.

If there is an $x \geq 0$ with $Ax \leq -1$, then that $x$ satisfies with $Ax \leq -1 < 0$.

If there is an $x \geq 0$ with $Ax < 0$ then assume such an $x$ exists with $Ax = (-a_1, -a_2, \ldots, -a_m)^T$. Let $a = \min \{a_1, a_2, a_3, \ldots, a_m\}$. Thus $a > 0$. Then $A(1/a)x = (-a_1/a, -a_2/a, \ldots, -a_m/a)^T \leq -1$ and $1/a x \geq 0$.

b) We set up a primal dual pair.

$$\begin{align*}
\text{max} & \quad 0 \cdot x \\
\text{primal P:} & \quad Ax \leq -1 \\
x & \geq 0 \\
\text{min} & \quad -1 \cdot y \\
\text{dual D:} & \quad A^T y \geq 0 \\
y & \geq 0 \\
z & \geq 0
\end{align*}$$

We have two statements:

i) there exists an $x \geq 0$ with $Ax < 0$,

ii) there exists $y \geq 0, z \geq 0$ with $A^T y \geq 0$ and $y \neq 0$

Our primal $P$ is bounded (value of objective function at most 0) and so there are two cases by Fundamental Theorem of LP.

Case 1. Assume that the primal is infeasible.

The dual is feasible ($y = 0$ works) so by the Fundamental Theorem of Linear Programming, the dual is either unbounded or has an optimal solution. But if the dual has an optimal solution, then by Strong Duality, we deduce that the primal has an optimal solution $x$ which
is feasible which is a contradiction. Thus the dual is unbounded and so we can find a feasible $y$ (i.e. $A^T y \geq 0$) with $-1 \cdot y < 0$ and so $y \neq 0$ and hence ii) holds.

The primal is infeasible and so by a) we have that there can be no $x \geq 0$ with $Ax \leq -1$ which means by our argument in part a) that there is no $x \geq 0$ with $Ax < 0$ and we conclude i) doesn’t hold.

Case 2. Assume the primal has an optimal solution $x^*$.

We apply a) again to note that if there is a feasible solution to the primal then there is an $x \geq 0$ with $Ax < 0$. Thus i) holds

Also, by Weak Duality, any feasible solution to the dual (i.e. any $y$ with $A^T y \geq 0, y \geq 0$) has $-1 \cdot y \geq 0 \cdot x^* = 0$. But $-1 \cdot y \geq 0$ and $y \geq 0$ implies that $y = 0$ and so ii) doesn’t hold.

Since Case 1 and 2 exhaust all possibilities we know that either i) or ii) holds but not both.