1. Consider an LP which has as its first constraint
\[-x_1 + x_2 - x_3 - x_4 \leq 1.\]
Assume that you know (based on other constraints) that $x_2 = 0$. Show that the dual variable $y_1$ associated with the first constraint is zero.

2. Consider the following LP:

\[
\begin{align*}
\text{max} & \quad c_1 x_1 + 7x_2 + 7x_3 + 11.1x_4 \\
(10 + d_1)x_1 & + 4.5x_2 + 1.2x_3 + 9x_4 \leq 100 \\
(10 + d_2)x_1 & + 4x_2 + 1x_3 + 8.2x_4 \leq 100 \\
(10 + d_3)x_1 & + 3x_2 + 3x_3 + 3x_4 \leq 100 \\
(10 + d_4)x_1 & + 2x_2 + 4x_3 + x_4 \leq 100 \\
(10 + d_5)x_1 & + x_2 + 5x_3 + x_4 \leq 100 \\
\end{align*}
\]

where $d_1d_2d_3d_4d_5$ are the first 5 digits of your student number.

a) Graph (sketch) the optimal value of the objective function as a function of $c_1$ for all $c_1 \in (-\infty, \infty)$. Use the LINDO package or any other software but provide a printout of at least one input and one output. Begin with $c_1 = 0$ and use ranging to determine an interval for $c_1$ for which you know the answer. Ranging gives you the interval in which the optimal basis $B$ is unchanged and the value for $x_1$ in the optimal solution gives the slope (Why?). Now choose a $c_1$ outside this interval and repeat. Continue until you know the optimal values for all possible $c_1$. You might need as many as nine intervals or as few as two intervals depending on your student number.

b) Consider the optimal value of the objective function as a function of the value $c_1$, say $f(c_1)$. Show that $f(c_1)$ is a concave upwards function.

3. (from an old exam) If you are given an optimal primal solution $x^*$ to an LP and you wish to deduce an optimal dual solution $y^*$, then you might try to determine $y^*$ using

1. Complementary Slackness of $y^*$ with $x^*$.
2. $y^*$ satisfies constraints (including positivity constraints if any) in dual, i.e. $y^*$ is a feasible solution of the dual.

Many of our examples in class and quizzes yielded unique optimal $y^*$ but in general there may be many optimal dual solutions. Are all $y$ satisfying (1),(2) optimal to the dual? Is every optimal dual solution determined as a solution to (1),(2)? Explain.

4. Let $A$ be an $m \times n$ matrix with $A \geq 0$ and each column of $A$ has a non-zero positive entry. Let $b \geq 0$. Then show that the LP

\[
\begin{align*}
\text{max} & \quad c \cdot x \\
Ax & \leq b \\
x & \geq 0 \\
\end{align*}
\]

always has an optimal solution.
a) Show there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ if and only if there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} \leq -1$.

Note: we use the definition $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_1 < y_1, x_2 < y_2, \ldots$ and $x_n < y_n$. This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol $>$ would generally mean $\geq$ and $\neq$ but this is not true for matrices or vectors. A vector $\mathbf{x}$ might satisfy $x \geq 0$ and also $\mathbf{x} \neq \mathbf{0}$ and yet still have some 0 entries. Such a vector $\mathbf{x}$ with 0 entries has $\mathbf{x} \not> \mathbf{0}$.

b) Let $A$ be an $m \times n$ matrix. Prove that either:

i) there exists an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ or

ii) there exists $\mathbf{y} \geq \mathbf{0}$ with $A^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$

but not both.

Hint: Extend the idea in a) and use it in setting up a primal dual pair.