The Inertia Tensor and After Dinner Tricks

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Abstract

After an introductory level discussion of momentum and angular momentum, the inertia tensor is introduced. The following theorem is proved: If I_1, I_2, and I_3 are the moments of inertia about the corresponding principle axes of rotation x_1, x_2, and x_3 such that $I_1 > I_2 > I_3$ then the x_1 and x_3 axes are stable axes of rotation while the x_2 axis is an unstable axis of rotation.

This paper was inspired by a lecture given by Professor Bill Heidbrink of the University of California, Irvine Department of Physics and Astronomy. If you would like to print out this document in full you should use the non-fragmented version . The original documents were writen in LaTeX and if you prefer you can get the original LaTeX source code. If you have any questions you can email me sdrasco@uci.eduand I would be *more* than happy to help out.

1 Momentum

One of the most fundamental concepts in physics is momentum. When you push a block across a table physicists say you have given the block momentum. For a single particle of mass m and velocity $\mathbf{v}$ it's momentum $\mathbf{p}$ is given by

$$\mathbf{p} = m\mathbf{v} \tag{1}$$

So you can see that when you push a block you give it some velocity which in turn means it has momentum.

One of the most useful properties of momentum is that if there are no forces acting on the particle inertia will be constant in time. To see why consider the famous law discovered by Isaac Newton

$$\mathbf{F} = m\mathbf{a} \tag{2}$$

This law can be written

$$\begin{aligned}\mathbf{F} &= m\frac{d^2\mathbf{x}}{dt^2} \\ &= \frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt}\right) \\ &= \frac{d\mathbf{p}}{dt}\end{aligned}$$

This version of Newton's second law is more compact[1]. Now suppose we don't have any forces acting. That means $d\mathbf{p}/dt = 0$ which of course means that $\mathbf{p}$ is constant in time. The importance of the conservation of momentum can not be overemphasized.

2 Angular momentum

If an particle has a lot of momentum and you want to drastically alter the path it is traveling along you have to apply a lot of force to the particle. You can see this from $\mathbf{F} = d\mathbf{p}/dt$. If you were to apply only a small force, the momentum would only change a small amount. This is one of the handiest ways to think of momentum. You ask yourself how hard it would be to drastically alter the motion of an object and you get an idea of how much momentum the object has. So an ice cube sliding along the table would have less momentum than a dictionary sliding along at the same speed.

Now suppose you were to take a heavy bowling ball and give it a good fast spin. You're not rolling it anywhere—just let it spin in place. You can imagine that it takes a substantial amount of *force* to get it to stop spinning. That is to say leaning down and blowing on it wouldn't be very effective. By the same logic as before you might think this means that the object has a lot of momentum. But the bowling ball doesn't have any velocity which means momentum (as we have defined it) is zero![2]

Low and behold physicists have another quantity called angular momentum to describe spinning objects such as our bowling ball. Angular momentum is defined as follows

$$\mathbf{L} = I\omega \tag{3}$$

where ω is the angular velocity of the object about one of the principle axes of the object[3] and I is something called the moment of inertia about that axis.

[1]You might be interested to know that Newton's second law actually *is* $\mathbf{F} = d\mathbf{p}/dt$. There is a big difference between this statement and equation (1) when the mass of the object of interest is a function of time.

[2]If you are substantially clever you might want to say that each chunk of the ball has some velocity of it's own as it goes round and round. This is a wise idea except the problem is that for each chunk of the ball having some velocity there is an equally size chunk on the other side of the ball with a velocity exactly opposite to the first chunk so the total momentum (as we have defined it) is still zero.

[3]I will explain what is meant by principle axes later.

I depends on the geometry of the object relative to the axis of rotation. I is defined as

$$I = \int \rho \left|\mathbf{r}\right|^2 d\tau$$

where ρ is mass density function, $d\tau$ is the differential volume element, and $\mathbf{r}$ is the shortest vector which points from the axis of rotation to the volume element.

Actually when we discuss a "force" which changes the way an object rotates we call it a torque rather than a force. The variable I will use for torque is $\mathbf{N}$. Just as when there are no forces acting on an object momentum is conserved—when there are no *torques* acting on an object *angular momentum* is conserved. This result is derived from the following equation which I will state without proof.

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} \tag{4}$$

To derive the law of conservation of angular momentum, set $\mathbf{N}$ equal to zero in equation 4 and apply the same argument we used to get the law of conservation of momentum. Again it must be said that the law of conservation of angular momentum can not be over emphasized.

You should notice the similarity between $\mathbf{F} = d\mathbf{p}/dt$ and $\mathbf{N} = d\mathbf{L}/dt$. The same similarity exists between $\mathbf{L} = I\omega$ and $\mathbf{p} = m\mathbf{v}$. These equations are *almost* perfect analogs except there is this funny business about I depending on what principle axis the object is rotating about. We want to develop a more complete equation for $\mathbf{L}$ which is a *perfect* analog to $\mathbf{p}$.

3 The inertia tensor

Consider a rectangular block of wood of uniform mass density. If you use center of mass of the block for the origin of your coordinate system then each of the three principle axes is perpendicular to two different faces of the block. I'm telling you this—it was *not* intended to be an obvious fact. For notation we will use (x_1, x_2, x_3) to denote the position of a point in this system of coordinates. Now for each of the axes we have a moment of inertia

$$I_1 = \int \rho \left|\mathbf{r}_1\right|^2 d\tau$$
$$I_2 = \int \rho \left|\mathbf{r}_2\right|^2 d\tau$$
$$I_3 = \int \rho \left|\mathbf{r}_3\right|^2 d\tau$$

If we make a matrix $\mathcal{I}$ defined by

$$\mathcal{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{5}$$

then we can write an equation for the angular momentum for any kind of crazy rotation of our block

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\mathbf{L} = \mathcal{I}\omega$$

Take a moment to think over what we just did until you can see that it agrees with equation 3.

You should know that $\mathcal{I}$ is not *just* a matrix in the same way that (L_1, L_2, L_3) is not *just* a matrix. In fact (L_1, L_2, L_3) is just a nice way to represent a real geometrical object—the angular momentum vector. It doesn't matter what crazy coordinate system you want to use to look at $\mathbf{L}$. It is a geometrical object. Similarly $\mathcal{I}$ is a geometrical object called a *tensor*. There isn't any easy way to draw a tensor with arrows on a graph as you would a vector, but this does not change the fact that a tensor *is* a geometrical object independent of your choice of coordinates.

4 Principle axes

Now I must have annoyed some people back there when I mentioned principle axes. I didn't even say what they are! I feel as though I have to say something about them.

I will state without proof here that the inertia tensor has the form

$$\mathcal{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where

$$I_{ij} = \int \rho \left(\delta_{ij} \textstyle\sum_k x_k^2 - x_i x_j \right) d\tau$$

and k runs from 1 to 3. The symbol δ_{ij} is the Kronecker delta symbol and is defined such that

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Now the wonderful thing is[4] that the eigenvectors of $\mathcal{I}$ are the principle axes for the frame you chose and the eigenvalues are the corresponding moments of inertia I_1, I_2, and I_3. So if $\mathbf{L} = (L_1', L_2', L_3')$ and $\omega = (\omega_1', \omega_2', \omega_3')$ are the angular momentum and angular velocity vectors as viewed from the principle axis coordinate system then we have

$$\begin{pmatrix} L_1' \\ L_2' \\ L_3' \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix}$$

[4] Again this is not supposed to be an obvious fact. You will have to take my word for it.

This is of course still just $\mathbf{L} = \mathcal{I}\omega$ but now we have a nice diagonal matrix to work with, we have principle axes of rotation, and we are very happy. From now on we will only use the center of mass for our origin and we will only use the principle axes coordinate system.

5 Stable and unstable rotations

It's now time for some informal definitions.

Definition 1 *If you give an object a twist about an axis and it seems to whirl around nicely about the axis you originally spun it on, we say that this axis is a **stable axis of rotation**.*

Definition 2 *If you give an object a twist about an axis and the object seems to not only rotate about the axis you spun it on but it starts to develope rotations about other axes, we say the axis it was originally spun on is an **unstable axis of rotation**.*

Theorem 3 *If I_1, I_2, and I_3 are the moments of inertia about the corresponding principle axes of rotation x_1, x_2, and x_3 such that $I_1 > I_2 > I_3$ then the x_1 and x_3 axes are stable axes of rotation while the x_2 axis is an unstable axis of rotation.*

I give the following informal proof of the theorem.

Proof. Assume, as described in the theorem I_1, I_2, and I_3 are the moments of inertia about the principle axes of rotation and $I_1 > I_2 > I_3$. The kinetic energy of a particle is given by the familiar equation $T = \frac{1}{2}mv^2$. I will state without proof that analogously for angular motion, kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}\mathcal{I}\left|\omega\right|^2 \\ &= \frac{1}{2}\mathcal{I}\left(\omega \cdot \omega\right) \\ &= \frac{1}{2}\left(\mathcal{I}\omega\right) \cdot \omega \\ &= \frac{1}{2}\mathbf{L} \cdot \omega \end{aligned}$$

Since $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ and of course $\omega = (\omega_1, \omega_2, \omega_3)$ we can write the above result as

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \tag{6}$$

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \tag{7}$$

Also we have

$$\left|\mathbf{L}\right|^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{8}$$

For reasons you will understand later—we are going to consider the quantity

$$2TI_1 - |\mathbf{L}|^2$$

It is important to understand that this quantity is equal to a constant. To see why look at the individual Terms. Of course 2 is a scalar constant. Since I_1 depends only on the geometry of the object it too is a constant in time. The potential energy of our object is approximately constant, therefore since energy is conserved[5] the kinetic energy T must be constant as well. Finally since there are no torques acting on our object the angular momentum is conserved. Thus $|\mathbf{L}|^2$ is constant in time.

Now we are all set to consider individual rotations. First suppose we give our object a spin so that $\omega_2 = \omega_3 \simeq 0$ and $|\omega| \simeq \omega_1$. Then we can write

$$\begin{aligned} 2TI_1 - |\mathbf{L}|^2 &= I_1\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - \left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right) \\ &= I_1 I_2\omega_2^2 + I_1 I_3\omega_3^2 - I_2^2\omega_2^2 - I_3^2\omega_3^2 \\ &= I_2\left(I_1 - I_2\right)\omega_2^2 + I_3\left(I_1 - I_3\right)\omega_3^2 \end{aligned}$$

but $\omega_2 = \omega_3 \simeq 0$ so we have

$$\begin{aligned} 2TI_1 - |\mathbf{L}|^2 &= I_2\left(I_1 - I_2\right)\omega_2^2 + I_3\left(I_1 - I_3\right)\omega_3^2 \\ &\simeq 0 = \text{constant} \end{aligned}$$

or

$$I_2\left(I_1 - I_2\right)\omega_2^2 + I_3\left(I_1 - I_3\right)\omega_3^2 \simeq 0 = \text{constant} \tag{9}$$

Similarly if we were to have set things up initially so that $\omega_1 = \omega_2 \simeq 0$ and $|\omega| \simeq \omega_3$ we would find

$$I_1\left(I_3 - I_1\right)\omega_1^2 + I_2\left(I_3 - I_2\right)\omega_2^2 \simeq 0 = \text{constant} \tag{10}$$

and if we set things up so that initially $\omega_1 = \omega_3 \simeq 0$ and $|\omega| \simeq \omega_2$ we would find

$$I_1\left(I_2 - I_1\right)\omega_1^2 + I_3\left(I_2 - I_3\right)\omega_3^2 \simeq 0 = \text{constant} \tag{11}$$

You may not realize it but that's it—we've proved the theorem! To understand why focus on equation 9. Recall that we assumed $I_1 > I_2 > I_3$ so both the terms in parentheses are positive. Also of course ω_2^2 and ω_3^2 are positive. This means that there can be no sizable changes in either ω_2 or ω_3. For if ω_2 got bigger the first term of the equation would increase. That means that the second term has to get smaller so that the total is still about 0. The best the second term can do is go to zero so we can't allow sizable changes in ω_2. By the same argument, there will be no sizable changes in ω_3. Thus if we only start off with rotation about x_1 no significant rotations can develop about the other axes. We conclude x_1 is a stable axis of rotation.

[5] Ignore any frictional forces.

The argument is almost exactly the same for equation 10. The difference here is that both terms in parentheses are negative. Regardless there can be no big changes in either term because the most the other term can do to compensate is go to zero. Thus if we only start off with rotation about x_3 no significant rotations can develop about the other axes. We conclude x_3 is a stable axis of rotation.

Finally look at equation 11. The first term in parentheses is negative while the second term in parentheses is positive. This means that we *can* allow significant changes in either ω_1 or ω_3. Thus if we only start off with rotation about x_2 rotations about the other axes *can develop*. We conclude x_2 is an *unstable* axis of rotation. ■

6 An after dinner trick

Now it is time for us to reward ourselves for our hard work. Since the subject matter here is physics we are able to *see* the results of our work animated in real time without the help of any fancy computers. All you need is a rigid rectangular object. An old textbook will do just fine[6]. Now what we want to do is figure out what are the principle axes and what are the relative magnitudes of the moments of inertia. You probably have in mind some integrals and diagonalization of matrices etc. Lucky for us we don't *really* need to fuss with these. I already told you what the principle axes are for a rectangular block when you pick the center of mass as your origin. Also it is not too hard to *eyeball* the relative magnitudes of the moments of inertia. We will go through it slowly so that you can see how it's done.

The mass density function of a book is approximately constant. That is if you don't have any lead inserts or bookmarks you will be safe to write

$$I_1 = \rho \int |\mathbf{r}_1|^2 \, d\tau$$

$$I_2 = \rho \int |\mathbf{r}_2|^2 \, d\tau$$

$$I_3 = \rho \int |\mathbf{r}_3|^2 \, d\tau$$

We need to name our axes. Call the axis that pierces both the front and back cover of the book x_1. Call the axis that runs perpendicular to the spine of the book and parallel to the covers x_2. Finally call the axis that runs parallel to the spine of the book x_3.

Lets compare I_1 and I_3. If you look at the axes for a while you will see on average a little chunk of the book is father from x_1 than x_3. Next look at the equations for I_1 and I_3. If chunks of the book are on average farther from x_1 than x_3 then you should see the result is $I_1 > I_3$. For the equations depends on

[6]Note, you need to use a rectangular textbook. No square textbooks will work. Ask yourself afterward why this is so.

the square of the distance of each little chunk. Really convince yourself of this before going on. It's essential that you agree with me here. It may help to know that the equations for I about a principle axis for a discrete mass distribution (a scattered bunch of n particles say) is given by

$$I = \textstyle\sum_n m_n r_n^2$$

Now that you have convinced yourself that $I_1 > I_3$ a similar analysis should convince you that $I_3 > I_2$. The fact that on average a little chunk of the book is father from x_3 than x_2 is not as obvious as the previous case. It may take a little longer to convince yourself of this.

After going through the above you should be a firm believer that if you really carried out the integrals you would find

$$I_1 > I_2 > I_3$$

Looking back at the theorem we just proved then this means it should be easy to make it so the book rotates purely about either x_1 or x_3. Also it means that you should have a heck of a time trying to get a pure rotation about x_2—it can be done—but you need to have just the right initial conditions and it is virtually impossible. At this point if you aren't wildly throwing books around the room[7] you must not have been paying attention. Give it a try!

Now you can try to do make guesses about the principle axes and relative magnitudes of moments of inertia are for all sorts of objects. Just make sure when you test your theories you don't injure yourself. You can now impress all your friends by challenging them to make stable rotations about unstable axes of everyday objects.

[7] You might want to strap a rubber band around the book or tape it shut to make sure the covers don't flap like wings as you toss the book.