1. [20 marks] Consider a $4 \times 6$ matrix $A$

\[
A = \begin{bmatrix}
0 & 1 & 2 & 1 & 1 & 1 \\
0 & 2 & 4 & 3 & 2 & 3 \\
0 & 2 & 4 & 3 & 3 & 5 \\
0 & 1 & 2 & 2 & 2 & 4
\end{bmatrix}
\]

There is an invertible matrix $E$ so that

\[
EA = \begin{bmatrix}
0 & 1 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

a) [3 marks] What is $\text{rank}(A)$?
b) [8 marks] Give a basis for the row space of $A$ and a basis for the column space of $A$.
c) [4 marks] Give a basis for the null space($A$).
d) [5 marks] Consider a vector $c \in \mathbb{R}^4$ so that $Ax = c$ is consistent (i.e. the system of equations has a solution). Let $A' = [c|A]$, i.e. $A'$ is the $4 \times 7$ matrix with $c$ being the first column and the remainder being the columns of $A$. What is $\text{rank}(A')$?

2. [15 marks] For the matrix

\[
A = \begin{bmatrix}
1 & -2 \\
2 & 1
\end{bmatrix}
\]

determine explicit matrices $M, D, M^{-1}$ where $D$ is a diagonal matrix, so that $A = MDM^{-1}$.

3. [20 marks]

Let $u_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

NOTE: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}$

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation satisfying

\[
T(u_1) = u_2, \quad T(u_2) = 2u_3, \quad T(u_3) = 3u_1.
\]

a) [5 marks] Give the matrix representation of $T$ with respect to the basis $\{u_1, u_2, u_3\}$.
b) [10 marks] Give the matrix representation of $T$ with respect to the basis $\{e_1, e_2, e_3\}$ (the standard basis). Give the explicit matrix with integer entries.
c) [5 marks] Give the matrix representing $T^3$ with respect to the basis $\{u_1, u_2, u_3\}$. What is the matrix representing $T^3$ with respect to the standard basis $\{e_1, e_2, e_3\}$? Again, you should be able to give explicit matrices.

4. [10 marks]

a) [5 marks] Explain why $\text{rank}(A) = \text{rank}(A^T)$.
b) [5 marks] Assume $A$ is diagonalizable. Show that $A$ is similar to $A^T$.

5. [5 marks] Show that the functions $e^x$, $e^{2x}$ and $x(x - 1)$ are linearly independent.

6. [10 marks] Let $A$ be a $3 \times 3$ diagonalizable matrix with $\det(A - \lambda I) = -(\lambda - 2)(\lambda - 3)(\lambda - 4)$. What are the eigenvalues of $A - 4I$?
7 [10 marks] Let \( A \) be a nonzero \( n \times n \) matrix satisfying \( A^k = 0 \). Show that \( A \) is not diagonalizable.

8 [10 marks] Let \( A \) be an \( n \times n \) matrix. Let \( V \) be the set of all matrices which are polynomials in \( A \) (of degree at most 3):

\[
V = \{a_3A^3 + a_2A^2 + a_1A + a_0I : a_3, a_2, a_1, a_0 \in \mathbb{R}\}
\]

a) [4 marks] Show that \( V \) is a subspace of the vector space of all \( n \times n \) matrices.

b) [6 marks] Assume that \( A^2 = A + 2I \). Show that \( \dim(V) \leq 2 \). Can \( \dim(V) = 0? \)