MATH 223: Row Space, Column Space and Rank of a matrix.

Let $A$ be an $m \times n$ matrix. Each column is a vector in $\mathbb{R}^m$ and each row, when interpreted as a column, is a vector in $\mathbb{R}^n$. Let $A_i$ denote the $i$th column of $A$. We define the column space of $A$, denoted $\text{colsp}(A)$ as the span{$A_1, A_2, \ldots, A_n$}. Similarly we define the row space of $A$, denoted $\text{rowsp}(A)$ as the span of the rows of $A$, when interpreted as column vectors in $\mathbb{R}^n$.

We have already noted that for $x = (x_1, x_2, \ldots, x_n)^T$, we have $Ax = \sum_{i=1}^n x_i A_i \in \text{colsp}(A)$. A consequence is that $\text{colsp}(A) = \text{Im}(f)$ where we use $\text{Im}(f)$ to denote the image space (or range) of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = Ax$.

We have previously noted the following

**Proposition 1** Let $A$ be an $m \times n$ matrix.
(a) If $M$ is an $m \times m$ matrix then \( \{ x : Ax = 0 \} \subseteq \{ x : MAx = 0 \} \)
(b) If $M$ is an invertible $m \times m$ matrix, then \( \{ x : Ax = 0 \} = \{ x : MAx = 0 \} \)

We proved (b) at the beginning of the course (in the context of \( \{ x : Ax = b \} \)) but you can specialize to $b = 0$. Results related to (a) were being used in the practice Midterm 1 in question 7.

We can also prove results for $\text{rowsp}(A)$ by simply using $\text{rowsp}(A) = \text{colsp}(A^T)$ but it makes sense to use the staircase pattern obtained by applying Gaussian elimination to $A$.

**Proposition 2** Let $A$ be an $m \times n$ matrix.
(a) If $M$ is an $m \times m$ matrix then $\text{rowsp}(MA) \subseteq \text{rowsp}(A)$
(b) If $M$ is an invertible $m \times m$ matrix, then $\text{rowsp}(MA) = \text{rowsp}(A)$

Consider the following example which we imagine was obtained by Gaussian elimination.

$$A = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 4 & -4 & 0 & 4 & 3 & 2 & 2 \\ 2 & -1 & 3 & 4 & 1 & 1 & 2 \\ 2 & 0 & 6 & 6 & 2 & 4 & 8 \end{bmatrix}$$

With $E$ invertible we obtain

$$EA = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any linear dependence among the columns such as $y_1A_1 + y_2A_2 + \cdots + y_n A_n = 0$ with $y = (y_1, y_2, \ldots, y_n)^T$ yields a solution to $Ay = 0$ and vice versa namely any $y = (y_1, y_2, \ldots, y_n)^T$ with $Ay = 0$ yields $y_1A_1 + y_2A_2 + \cdots + y_n A_n = 0$. Let $I$ denote a subset of $\{1, 2, \ldots, n\}$, namely a subset of the column indices. Let $A_i$ denote the $i$th column of $A$ so that $(EA)_i$ denotes the $i$th column of $EA$. We deduce the following using Proposition 1.

**Proposition 3** Let $A, E$ be given with $E$ being invertible. The set of columns $\{ A_i : i \in I \}$ is linearly dependent if and only if the set of columns $\{ (EA)_i : i \in I \}$ is linearly dependent.
Theorem 4 Let \( A, E \) be given with \( E \) being invertible. It then follows that the set of columns \( \{ A_i : i \in I \} \) is linearly independent if and only if the set of columns \( \{ (EA)_i : i \in I \} \) is linearly independent and hence the set of columns \( \{ A_i : i \in I \} \) forms a basis for \( \text{colsp}(A) \) if and only if the set of columns \( \{ (EA)_i : i \in I \} \) forms a basis for \( \text{colsp}(EA) \).

When we look at staircase patterns \( EA \), where \( E \) is invertible, it is easy to identify linearly independent columns of \( EA \) whose span is \( \text{colsp}(EA) \). Given that the sets of columns that are linearly dependent in \( A \) are precisely those that are linearly dependent in \( EA \), then it is also true that those that are linearly independent in \( A \) are precisely those that are linearly independent in \( EA \). Hence a set of columns of \( A \) yielding a column basis for \( \text{colsp}(A) \) will correspond to a set of columns of \( EA \) yielding a column basis for \( \text{colsp}(EA) \). Note that the idea is that the 1st, 2nd and 5th columns of \( EA \) yield a column basis for \( \text{colsp}(EA) \) if and only if the 1st, 2nd and 5th columns of \( A \) yield a column basis for \( \text{colsp}(A) \). It is straightforward to deduce that a basis for \( \text{colsp}(EA) \) are columns 1, 2 and 5:

\[
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

and so, by Corollary 4, a basis for \( \text{colsp}(A) \) is

\[
\begin{bmatrix}
2 \\
4 \\
2
\end{bmatrix}
\begin{bmatrix}
-2 \\
-4 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\]

There are other choices for column bases but it is easiest to chose the columns of \( A \) whose corresponding columns in \( EA \) contain the pivots.

We can now use the (relatively) easy observation that the nonzero rows of \( EA \) form a basis for \( \text{rowsp}(EA) \). namely a basis for \( \text{rowsp}(EA) \) is \( \{(2, -2, 0, 2, 1, 0, 0)^T, (0, 1, 3, 2, 0, 2, 1)^T, (0, 0, 0, 0, 1, 3, 2)^T\} \). Combine this with Proposition 2 with \( E \) being invertible and we have that the nonzero rows of \( EA \) are also a basis for \( \text{rowsp}(A) \).

We have defined \( \text{rowsp}(A) = \text{span}\{(2, -2, 0, 2, 1, 0, 0)^T, (4, -4, 0, 4, 3, 2, 2)^T, (2, -1, 3, 4, 1, 1, 3)^T, (2, 0, 6, 6, 2, 4, 8)^T\} \). With \( E \) being invertible we have \( \text{rowsp}(A) = \text{rowsp}(EA) \) and so a basis for \( \text{rowsp}(A) \) is \( \{(2, -2, 0, 2, 1, 0, 0)^T, (0, 1, 3, 2, 0, 1, 3)^T, (0, 0, 0, 0, 1, 3, 2)^T\} \). Please note that \( E \) being invertible does not mean that the first 3 rows of \( A \) form a basis for \( \text{rowsp}(A) \), although it is possible.

Theorem 5 \( \text{dim}(\text{rowsp}(A)) = \text{dim}(\text{colsp}(A)) \),

Proof: We have \( \text{dim}(\text{rowsp}(A)) \) being equal to the number of non zero rows of \( EA \) and hence the number of pivots and we have \( \text{dim}(\text{colsp}(A)) \) being equal to the size of a basis for \( \text{colsp}(EA) \) which is the number of pivots. 

Thus Theorem 5 allows us to define

\[
\text{rank}(A) = \text{dim}(\text{colsp}(A)) = \text{dim}(\text{rowsp}(A)).
\]

From this we obtain the following lovely result

Theorem 6 Let \( A \) be an \( m \times n \) matrix. Then \( \text{rank}(A) + \text{dim}(\text{nullsp}(A)) = n \).

Proof: \( \text{dim}(\text{nullsp}(A)) \) is the number of free variables. We have the number of pivot variables and the number of free variables is \( n \).