

MATH 223: A Putnam Problem using our Linear Algebra.

(This problem came from a 1994 Putnam Problem.)

Problem: Let A, B , be two integer 2×2 matrices (i.e. 2×2 matrices with integer entries). Assume they have the property that $A, A + B, A + 2B, A + 3B, A + 4B$ which are integer 2×2 matrices all with integer 2×2 inverse matrices. We are asked to show that $A + 5B$ has an integer inverse.

Proof: An integer 2×2 matrix C has an integer inverse if and only if $\det(C) = \pm 1$. This isn't too hard to prove using our expression for the inverse.

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad C^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

Since C has integer entries, $\det(C)$ is an integer. In order for C to have an integer inverse, we would need $ad - bc$ to divide all of a, b, c, d . Let $ad - bc = e$. Then we may write $a = ep, b = eq, c = er, d = es$, for some integers p, q, r, s . Then $\det(C) = e^2 \det\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)$ which is a contradiction unless $e^2 = \pm e$ and so $\det(C) = \pm 1$.

Our hypothesis that $A + iB$ has an integer inverse for $i = 0, 1, 2, 3, 4$ means $\det(A + iB) = \pm 1$ for $i = 0, 1, 2, 3, 4$.

Now $\det(A + \lambda B)$ is a quadratic expression in λ (write it out!). Moreover it takes on the values ± 1 for 5 values of λ and hence for three values of λ it takes on the same value; say 1 without loss of generality.

A quadratic expression in λ that takes on the same value for 3 choices of λ is a constant. You may know this fact from your knowledge of the functions $f(x) = ax^2 + bx + c$. It is actually a consequence of the deep theorem, called the Fundamental Theorem of Algebra. In our case we have $f(x_1) = f(x_2) = f(x_3)$ and so the quadratic function $g(x) = ax^2 + bx + (c - f(x_1))$ has three roots at x_1, x_2, x_3 and so must be the zero function. Hence $\det(A + \lambda B) = 1$ for all λ . But then $\det(A + 5B) = 1$ and so $A + 5B$ has an integer inverse.

The hypotheses are not vacuous. An example of a pair A, B would be

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$