ORTHOGONAL VECTOR SPACES

Let $U, V$ be vector spaces with $U \subseteq V$. We consider

$$U^\perp = \{ v \in \mathbb{R}^n : \text{ for all } u \in U, < u, v > = 0 \}$$

**Theorem 0.1** $U^\perp$ is a vector space.

**Proof:** We show that $U^\perp$ is a vector space. Here we must verify that $0 \in U^\perp$ since this will not follow from the other two closure rules. We have $0 \in U^\perp$ because $< u, 0 > = 0$ always for any choice $u$. Also if $x, y \in U^\perp$, then $< x + y, u > = < x, u > + < y, u >$ and $< cx, u > = c < x, u >$ by our inner product axioms. Thus if for all $u \in U$, $< x, u > = 0$ and $< y, u > = 0$, then we conclude that $< x + y, u > = < x, u > + < y, u > = 0 + 0 = 0$ and also $< cx, u > = c < x, u > = c \cdot 0 = 0$. Thus we have $x + y$ and $cx$ in $U^\perp$, verifying closure. So $U^\perp$ is a vector space. 

Consider a vector space $U \subseteq \mathbb{R}^n$. Thus we are thinking of $V = \mathbb{R}^n$ with the standard basis $e_1, e_2, \ldots, e_n$. Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for $U$. Then if we write each $u_i$ with respect to the standard basis we can form a matrix $A = (a_{ij})$ with the $i$th row $A$ being $u_i^T$. Thus row space($A$) = $U$ and dim($U$) = rank($A$). Then

$$\text{null space}(A) = \{ x : A x = 0 \} = \{ x : < x, u_i > = 0 \text{ for } i = 1, 2, \ldots, k \}$$

$$= \{ x : < x, u > = 0 \text{ for all } u \in U \} = U^\perp$$

Here we are assuming $< x, u_i >$ is the standard dot product. Thus dim($U$) + dim($U^T$) = $n$ using our result that dim(nullsp($A$)) = rank($A$) = $n$ where $n$ is the number of columns in $A$.

These ideas will happily generalize to two vector spaces $U, V$ with $U \subseteq V$ with a general inner product. We do not need $V = \mathbb{R}^n$ but we can benefit from an orthonormal basis for $V$ in order to use the null space idea. If we apply Gram Schmidt or otherwise, we can obtain a basis $\{v_1, v_2, \ldots, v_n\}$ with the orthonormal properties:

$$< v_i, v_j > = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (*)$$

Now proceed much as before, expressing

$$u_i = \sum_{j=1}^{n} a_{ij} v_j$$

since $\{v_1, v_2, \ldots, v_n\}$ is a basis for $V$ and $u_i \in V$. Let $A$ be the associated $k \times n$ matrix. Now consider any vector $w \in V$ which we can write as $w = \sum_{j=1}^{n} w_j v_j$. Let $w$ denote the vector in the coordinates of the orthonormal basis so $w = (w_1, w_2, \ldots, w_n)^T$ Then

$$< u_i, w > = < \sum_{j=1}^{n} a_{ij} v_j, \sum_{\ell=1}^{n} w_\ell v_\ell >$$

$$= \sum_{j=1}^{n} a_{ij} \left( < v_j, \sum_{\ell=1}^{n} w_\ell v_\ell > \right)$$

$$= \sum_{j=1}^{n} a_{ij} \left( \sum_{\ell=1}^{n} w_\ell \left( < v_j, v_\ell > \right) \right)$$
\[ = \sum_{j=1}^{n} a_{ij}w_j \]

using properties of (*). Now \( \sum_{j=1}^{n} a_{ij}w_j \) is the \( i \)th entry of \( Aw \). Thus we have a way of expressing \( U^\perp \) as the null space(\( A \)) and we have the desired result.

**Theorem 0.2** Let \( U, V \) be vector spaces over \( \mathbb{R} \) with \( U \) a subspace of \( V \) and \( V \) is finite dimensional. Then \( \dim(U) + \dim(U^\perp) = \dim(V) \).

Another approach that doesn’t use an orthonormal basis of \( V \) (with respect to the given inner product) but just any basis \( v_1, v_2, \ldots, v_n \), we use the observation that for a given \( u_i \), the function \( <u_i, x> \) is a linear transformation \( V \rightarrow \mathbb{R} \) and so has an associated \( 1 \times n \) matrix. Now we verify that the \( k \) linear transformations \( <u_i, x> \) are linearly independent (and so the \( k \times n \) matrix formed by these rows has rank = \( k \)). Assume

\[ \sum_{i=1}^{k} c_i <u_i, x> \equiv 0 \]

where we use the notation \( \equiv 0 \) to mean the identically 0 function, namely the \( 0 \) vector in the space of functions. But now

\[ \sum_{i=1}^{k} c_i <u_i, x> = <\sum_{i=1}^{k} c_i u_i, x> \text{ for all } x \]

but when we evaluate the righthand side at \( x = \sum_{i=1}^{k} c_i u_i \), we obtain \( <x, x> = 0 \) and so by the axioms of an inner product we have \( x = 0 \) i.e. \( \sum_{i=1}^{k} c_i u_i = 0 \) which forces \( c_1 = c_2 = \cdots = c_k = 0 \) since the vectors \( u_1, u_2, \ldots, u_k \) are linearly independent.

**Theorem 0.3** Let \( U, V \) be vector spaces with \( U \) a subspace of \( V \) and \( V \) is finite dimensional. Then \( U^\perp \perp = U \).