MATH 223. Orthogonal Vector Spaces.

Let U, V be vector spaces with $U \subseteq V$. We consider

$$U^{\perp} = \{\mathbf{v} \in \mathbf{R}^n \ : \ \text{for all } \mathbf{u} \in U, \ < \mathbf{u}, \mathbf{v} >= 0\}$$

Theorem 0.1 *$U^{\perp}$ is a vector space.*

Proof: We have that U^T is a vector space. Here we must verify that $\mathbf{0} \in U^{\perp}$ since this will not follow from the two closure ideas. That is because $< \mathbf{u}, \mathbf{0} >= 0$ always. Also if $\mathbf{x}, \mathbf{y} \in U^{\perp}$, then $< \mathbf{x}+\mathbf{y}, \mathbf{u} >=< \mathbf{x}, \mathbf{u} > + < \mathbf{y}, \mathbf{u} >$ and $< c\mathbf{x}, \mathbf{u} >= c < \mathbf{x}, \mathbf{u} >$ by our inner product axioms. Thus if for all $\mathbf{u} \in U$, $< \mathbf{x}, \mathbf{u} >= 0$ and $< \mathbf{y}, \mathbf{u} >= 0$, then we conclude that $< \mathbf{x}+\mathbf{y}, \mathbf{u} >=< \mathbf{x}, \mathbf{u} > + < \mathbf{y}, \mathbf{u} >= 0 + 0 = 0$ and also $< c\mathbf{x}, \mathbf{u} >= c < \mathbf{x}, \mathbf{u} >= c \cdot 0 = 0$. Thus we have $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ in $U^{\perp}$, verifying closure. So $U^{\perp}$ is a vector space. ■

Consider a vector space $U \subseteq \mathbf{R}^n$. Thus we are thinking of $V = \mathbf{R}^n$ with the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a basis for U. Then if we write each $\mathbf{u}_i$ with respect to the standard basis we can form a matrix $A = (a_{ij})$ with the ith row A being $\mathbf{u}_i^T$. Thus row space$(A) = U$ and $\dim(U) = \text{rank}(A)$. Then

$$\begin{aligned} \text{null space}(A) &= \{\mathbf{x} \ : \ A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \ :< \mathbf{x}, \mathbf{u}_i >= 0 \text{ for } i = 1, 2, \ldots, k\} \\ &= \{\mathbf{x} \ :< \mathbf{x}, \mathbf{u} >= 0 \text{ for all } \mathbf{u} \in U\} = U^{\perp} \end{aligned}$$

Thus $\dim(U) + \dim(U^T) = n$ using our result that $\dim(nullsp(A)) + \text{rank}(A) = n$ where n is the number of columns in A These ideas will happily generalize to two vector spaces U, V with $U \subseteq V$. We do not need $V = \mathbf{R}^n$ but we need an orthonormal basis for V in order to use the null space idea. If we apply Gram Schmidt or otherwise, we can obtain a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ with the orthonormal properties:

$$< \mathbf{v}_i, \mathbf{v}_j >= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad (*)$$

Now proceed much as before writing

$$\mathbf{u}_i = \sum_{j=1}^{n} a_{ij}\mathbf{v}_j$$

since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V and $\mathbf{u}_i \in V$. Let A be the associated $k \times n$ matrix. Now consider any vector $\mathbf{w} \in V$ which we can write as $\mathbf{w} = \sum_{j=1}^{n} w_j\mathbf{v}_j$. Let $\mathbf{w}$ denote the vector in the coordinates of the orthonormal basis so $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ Then

$$\begin{aligned} < \mathbf{u}_i, \mathbf{w} >&=< \sum_{j=1}^{n} a_{ij}\mathbf{v}_j, \sum_{\ell=1}^{n} w_\ell\mathbf{v}_\ell > \\ &= \sum_{j=1}^{n} a_{ij} \left(< \mathbf{v}_j, \sum_{\ell=1}^{n} w_\ell\mathbf{v}_\ell > \right) \\ &= \sum_{j=1}^{n} a_{ij} \left(\sum_{\ell=1}^{n} w_\ell \left(< \mathbf{v}_j, \mathbf{v}_\ell > \right) \right) \\ &= \sum_{j=1}^{n} a_{ij} w_j \end{aligned}$$

using properties of (*). Now $\sum_{j=1}^{n} a_{ij}w_j$ is the ith entry of $A\mathbf{w}$. Thus we have a way of expressing $U^{\perp}$ as above and we have the desired result

Theorem 0.2 *Let U, V be vector spaces over $\mathbf{R}$. Then* $\dim(U) + \dim(U^T) = \dim(V)$.