MATH 223. Orthogonal Vector Spaces.

Let \( U, V \) be vector spaces with \( U \subseteq V \). We consider

\[
U^\perp = \{ v \in \mathbb{R}^n : \text{for all } u \in U, \ < u, v >= 0 \}
\]

**Theorem 0.1** \( U^\perp \) is a vector space.

**Proof:** We have that \( U^T \) is a vector space. Here we must verify that \( 0 \in U^\perp \) since this will not follow from the two closure ideas. That is because \( < u, 0 >= 0 \) always. Also if \( x, y \in U^\perp \), then

\[
< x + y, u >= < x, u > + < y, u > \quad \text{and} \quad < cx, u >= c < x, u >
\]

by our inner product axioms. Thus if for all \( u \in U, \ < x, u >= 0 \) and \( < y, u >= 0 \), then we conclude that \( < x + y, u >= 0 \) + \( < y, u >= 0 \) and also \( < cx, u >= c < x, u >= 0 \). Thus we have \( x + y \) and \( cx \) in \( U^\perp \), verifying closure. So \( U^\perp \) is a vector space. \( \blacksquare \)

Consider a vector space \( U \subseteq \mathbb{R}^n \). Thus we are thinking of \( V = \mathbb{R}^n \) with the standard basis \( e_1, e_2, \ldots, e_n \). Let \( \{ u_1, u_2, \ldots, u_k \} \) be a basis for \( U \). Then if we write each \( u_i \) with respect to the standard basis we can form a matrix \( A = (a_{ij}) \) with the \( i \)th row \( A \) being \( u_i^T \). Thus row space(\( A \)) = \( U \) and \( \dim(U) = \text{rank}(A) \). Then

\[
\text{null space}(A) = \{ x : Ax = 0 \} = \{ x : < x, u_i >= 0 \text{ for } i = 1, 2, \ldots, k \}
\]

\[
= \{ x : < x, u >= 0 \text{ for all } u \in U \} = U^\perp
\]

Thus \( \dim(U) + \dim(U^T) = n \) using our result that \( \dim(\text{nullsp}(A)) + \text{rank}(A) = n \) where \( n \) is the number of columns in \( A \) These ideas will happily generalize to two vector spaces \( U, V \) with \( U \subseteq V \).

We do not need \( V = \mathbb{R}^n \) but we need an orthonormal basis for \( V \) in order to use the null space idea. If we apply Gram Schmidt or otherwise, we can obtain a basis \( \{ v_1, v_2, \ldots, v_n \} \) with the orthonormal properties:

\[
< v_i, v_j > = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases} \quad (\star)
\]

Now proceed much as before writing

\[
u_i = \sum_{j=1}^{n} a_{ij} v_j
\]

since \( \{ v_1, v_2, \ldots, v_n \} \) is a basis for \( V \) and \( u_i \in V \). Let \( A \) be the associated \( k \times n \) matrix. Now consider any vector \( w \in V \) which we can write as \( w = \sum_{j=1}^{n} w_j v_j \). Let \( w \) denote the vector in the coordinates of the orthonormal basis so \( w = (w_1, w_2, \ldots, w_n)^T \). Then

\[
< u_i, w > = \sum_{j=1}^{n} a_{ij} v_j = \sum_{\ell=1}^{n} w_{\ell} v_{\ell} >
\]

\[
= \sum_{j=1}^{n} a_{ij} \left( < v_j, \sum_{\ell=1}^{n} w_{\ell} v_{\ell} > \right)
\]

\[
= \sum_{j=1}^{n} a_{ij} \left( \sum_{\ell=1}^{n} w_{\ell} ( < v_j, v_{\ell} > ) \right)
\]

\[
= \sum_{j=1}^{n} a_{ij} w_j
\]

using properties of (\( \star \)). Now \( \sum_{j=1}^{n} a_{ij} w_j \) is the \( i \)th entry of \( A w \). Thus we have a way of expressing \( U^\perp \) as above and we have the desired result

**Theorem 0.2** Let \( U, V \) be vector spaces over \( \mathbb{R} \). Then \( \dim(U) + \dim(U^T) = \dim(V) \).