Big new concepts in MATH 223 include a vector space, linear independence (or linear dependence), and dimension.

**Definition** A set \( S = \{v_1, v_2, \ldots, v_k\} \) of \( k \) vectors is said to be linearly dependent if there are coefficients \( a_1, a_2, \ldots, a_k \) not all zero such that \( a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \).

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Note that \( 0 \) is a linearly dependent set, since \( 1 \cdot 0 = 0 \).

These definitions are more symmetric than for example identifying \( S \) as linearly dependent of one vector in \( S \) is a linear combination of the others. Note however if \( v_i \) is a linear combination of the other vectors in \( S \), then \( \text{span}(S \setminus v_i) = \text{span}(S) \).

It makes some sense to choose a minimal subset \( S' \subseteq S \) with \( \text{span}(S') = \text{span}(S) \). Then \( S' \) must be linearly independent. You might note that the span of the empty set is naturally defined to be \( \{0\} \). Such boundary cases can be a bit awkward.

**Definition** For a vector space \( V \), a basis is a linearly independent set of vectors \( S \) so that \( \text{span}(S) = V \).

There would be two ways to find a basis. Either begin with a spanning set, and reduce if there are any dependencies. Alternatively build the basis from the ground up as a linearly independent set contained in \( V \).

**Theorem** Any basis for a vector space \( V \) has the same cardinality.

Proof: We let \( B_1 = \{u_1, u_2, \ldots, u_k\} \) and \( B_2 = \{v_1, v_2, \ldots, v_l\} \) be two bases for \( V \). Assume that \( l > k \). Now because \( B_1 \) is a basis, then any vector in \( V \) is a linear combination of vectors in \( B_1 \) and so we may write, without strange names for the coefficients, that

\[
v_j = \sum_{i=1}^{k} a_{ij} u_i
\]

Thus if we let \( A = (a_{ij}) \) be the matrix with these entries then the \( j \)th column of \( A \) corresponds to \( v_j \). Now because \( k < l \), then when we solve \( Ax = 0 \), we will have at most \( k \) pivot variables and hence at least \( l - k > 0 \) free variables and hence an \( x \neq 0 \) with \( Ax = 0 \).

We think of \( x \) as yielding a linear combination of the \( v_j \)'s yielding the zero vector, which would be a contradiction. Let \( x = (x_1, x_2, \ldots, x_l)^T \). Then

\[
\sum_{j=1}^{l} x_j v_j = \sum_{j=1}^{l} x_j \left( \sum_{i=1}^{k} a_{ij} u_i \right) = \sum_{i=1}^{k} \left( \sum_{j=1}^{l} a_{ij} \right) x_j = \sum_{i=1}^{k} 0 \cdot x_j = 0
\]

This has verified that \( B_2 \) is linearly dependent, a contradiction to \( B_2 \) being a basis and hence we conclude that \( k = l \).

**Definition** The dimension of a vector space \( V \) is the cardinality of any basis for \( V \).

The dimension of \( \mathbb{R}^t \) is \( t \) since we can identify a basis of \( \mathbb{R}^t \) as \( \{e_1, e_2, \ldots, e_t\} \) where \( e_i \) denote the vector with a 1 in the \( i \)th coordinate and 0’s elsewhere. Any vector space \( V \) contained in \( \mathbb{R}^t \),
has dimension at most $t$. (How should you show that the dimension is at most $t$? Assume you have $t + 1$ linear independent vectors in $U$ and derive a contradiction). Thus dimension is being used as a piece of mathematical terminology for vector spaces in the context of bases and does not refer some English meaning of dimension. Maybe we would have been better to have a separate term but this is not standard.