MATH 223. Equiangular lines in d dimensions.

Notes adapted from notes of Aart Blokhuis of Eindhoven.

Consider the problem of equiangular lines in euclidean space. Lines through the origin in $\mathbf{R}^d$ are one-dimensional subspaces and can be represented by $L =< \mathbf{u} >$ or $L = span(\mathbf{u})$, where $\mathbf{u}$ is one of the two unit vectors on line L. The angle $0 \leq \phi \leq \pi/2$ between two lines $< \mathbf{u} >$ and $< \mathbf{v} >$ is then determined by $\cos\phi = \alpha$, where $\alpha = |(u, v)|$.

A family of lines is called *equiangular* if every pair of them determines the same angle. We are interested in an upper bound for the number of equiangular lines in $\mathbf{R}^d$. This bound is due to Lemmens and Seidel, a polynomial argument is given by Koornwinder, the following "tensor"-argument is essentially equivalent.

So, let $L =< \mathbf{u} >$ be a line in $\mathbf{R}^d$, where vectors are column vectors. We can form the d by d matrix $U = \mathbf{u}\mathbf{u}^T$. This is a symmetric matrix with trace $\mathrm{tr}(U) = 1$ since $\mathbf{u}$ is chosen to be a unit vector. Note U is the projection matrix onto the line $< \mathbf{u} >$.

Now there must be a fixed number $0 < \alpha < 1$ for our collection of lines so for any two different lines given buy unit vectors $< \mathbf{u} >$, $< \mathbf{v} >$in our collection we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \alpha$.

Let the equiangular lines in $\mathbf{R}^d$ be given by $\{< v_i >: \ i = 1, 2, \ldots, t\}$. I claim the associated projection matrices are linearly independent. Assume the contrary, namely there are constants $c_1, c_2, \ldots, c_t$ not all zero so that

$$\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T = 0,$$

the righthand side being the $d \times d$ zero matrix. If we multiply on the right by $\mathbf{v}_j$ and on the left by $\mathbf{v}_j^T$ we obtain that

$$\mathbf{v}_j^T \left(\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T \right) \mathbf{v}_j = \mathbf{v}_j^T 0 \mathbf{v}_j$$

and so we have (for any choice of j)

$$\left(\sum_{i=1}^{t} c_i \right) \alpha^2 + (1 - \alpha^2) c_j = 0$$

using $\mathbf{v}_j^T \mathbf{v}_i \mathbf{v}_i^T \mathbf{v}_j = \alpha^2$ for $i \neq j$ and $\mathbf{v}_j^T \mathbf{v}_j \mathbf{v}_j^T \mathbf{v}_j = 1$ and with the righthand side being the 1×1 matrix 0. Since $\left(\sum_{i=1}^{t} c_i\right)$ is a constant independent of c_j and $1 - \alpha^2 \neq 0$, then c_j must be the same for each j. Thus $c_1 = c_2 = \cdots = c_t$. But now this yields a contradiction: For example

$$\mathrm{tr} \left(\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T \right) = \sum_{i=1}^{t} c_i = t c_1 \quad ,$$

and so $c_1 = c_2 = \cdots = c_t = 0$. This contradiction shows that the t matrices $\mathbf{v}_i \mathbf{v}_i^T$ are linearly independent.

The space of real symmetric $d \times d$ matrices is $d(d+1)/2$ dimensional (determined by entries below and on the diagonal; you did this on assignment 5). This gives us the upper bound $d(d+1)/2$ for the size of a set of equiangular lines in $\mathbf{R}^d$.

It is possible to compute α if the upper bound is realized. Problem: Show that $\alpha = 1/\sqrt{d+2}$.

How good is this bound? For $n = 0$ and $n = 1$ we have (formal) equality, but no interesting geometrical picture. For $n = 2$ we get three lines at $60°$ angles $(\pi/3)$ and $\alpha = 1/2$. For $n = 3$ the

bound is 6 with $\alpha = 1/\sqrt{5}$. This is realized by the 6 main diagonals of the icosahedron, or to be more explicit: let τ be the positive root of the quadratic $x^2 - x - 1 = 0$, so $\tau = (1+\sqrt{5})/2$. Now consider the six lines spanned by the 12 vectors $(0, \pm 1, \pm\tau)$, and their cyclic permutations.
The next case of equality is $n = 7$. Here the example (which is unique) is most easily described using the Fano plane, the projective plane of order 2. Consider the 28 lines spanned by the 56 vectors $(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0)$ and their cyclic permutations. Note that the 'support' of each vector corresponds to three collinear points in the Fano plane, a $2-(7,3,1)$ design. Since two lines in the Fano plane intersect in a unique point, all relevant inner products are ± 1.

People can also handle the case $d = 23$.

Generalizations to $\mathbf{C}^d$

Recently much attention has been given to the analogous problem in complex space $\mathbf{C}^d$. A complex line through the origin is again a one dimensional subspace $< \mathbf{u} >$, where we take $\mathbf{u}$ in such a way that $\mathbf{u}^H\mathbf{u} = 1$. $\mathbf{u}^H$ stands for the conjugate transpose of u. To define the 'angle' between two lines we compute $(u, v) = \mathbf{u}^H\mathbf{v}$ and take the absolute value $\alpha = |\mathbf{u}^H\mathbf{v}|$. As before we can prove an upper bound for the number of equiangular lines. Associate to a line $< \mathbf{u} >$ the d by d matrix $U = \mathbf{u}\mathbf{u}^H$. This is now a hermitian matrix, with trace 1. Hermitian matrices form a *real* vector space of dimension d^2. The dimension follows with two contributions for each off diagonal entry below the main diagonal ($\mathbf{C}$ has dimension 2 over $\mathbf{R}$) and the diagonal entries are all real. As before we get an upper bound, in this case d^2.
Problem: Fill in the details in the above 'proof' and show that in case of equality one has $\alpha = 1/\sqrt{n+1}$.
Problem: Construct a set of 4 equiangular lines in $\mathbf{C}^2$ and 9 in $\mathbf{C}^3$.
Interesting fact: It seems that for all d the bound can be reached, but nobody knows how or why. Sets realizing equality have several interesting names, one of them being SICPOVMS: see for instance:
http://info.phys.unm.edu/papers/reports/sicpovm.html
All known SICPOVMS have been constructed as follows: Consider the subgroup of $Gl(n, \mathbf{C})$ generated by a cyclic permutation of the coordinates, and $\mathrm{diag}(1, \zeta, \ldots, \zeta^{n-1})$ where ζ is a (primitive) n-th root of unity. This group has order n^3 but in its representation on the lines only n^2. For a suitable line $< \mathbf{u} >$ the orbit is a SICPOVM, at least it seems so numerically in dimensions up 45.
Have fun, and let me know if you find anything spectacular, Aart Blokhuis.