MATH 223: Eigenvalues and Eigenvectors with Diagonalization.

Consider a 2×2 matrix A. When a vector $\mathbf{v}$ satisfies

$$\mathbf{v} \neq \mathbf{0},$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say that $\mathbf{v}$ is an *eigenvector* of A of *eigenvalue* λ. We note

$$A(k\mathbf{v}) = k(A\mathbf{v}) = k(\lambda\mathbf{v}) = \lambda(k\mathbf{v}),$$

which says that non zero multiples of eigenvectors yield more eigenvectors of the same eigenvalue. Let us first consider the geometric transformations we previously mentioned. An eigenvector will correspond to a direction that is fixed (or reversed) by the transformation.

$$D(2,3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

will have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 2 and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigenvector of eigenvalue 3. The identity matrix I has the property that any non zero vector $\mathbf{v}$ is an eigenvector of eigenvalue 1.

The rotation matrix $R(\theta)$ has no eigenvectors, by the geometric reasoning that no directions are preserved, unless $\theta = 0, \pi$. There will be no (real) roots of the quadratic.

The *shear* matrix $G_{12}(\gamma) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 1 but no other eigenvectors (other than multiples) for $\gamma \neq 0$.

The following analysis is critical in seeking eigenvectors and eigenvalues:

$$\begin{aligned} &\text{there exists a } \mathbf{v} \text{ with } A\mathbf{v} = \lambda\mathbf{v}; \quad \mathbf{v} \neq \mathbf{0} \\ \text{if and only if } &\text{ there exists a } \mathbf{v} \text{ with } A\mathbf{v} = \lambda I\mathbf{v}; \quad \mathbf{v} \neq \mathbf{0} \\ \text{if and only if } &\text{ there exists a } \mathbf{v} \text{ with } (A - \lambda I)\mathbf{v} = \mathbf{0}; \quad \mathbf{v} \neq \mathbf{0} \\ \text{if and only if } &\det(A - \lambda I) = 0 \end{aligned}$$

Now

$$\begin{aligned} \det(A - \lambda I) = \det(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}) &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A), \end{aligned}$$

which (for 2×2 matrices) is a quadratic function in λ and whose roots you can seek by standard methods.

Sample computation

$$A = \det(\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix})$$

$$\det(A - \lambda I) = \det(\begin{bmatrix} .7-\lambda & .3 \\ 2 & -\lambda \end{bmatrix})$$

$$\begin{aligned} &= (.7-\lambda)(-\lambda) - .3\times 2 \\ &= \frac{1}{10}(10\lambda^2 - 7\lambda - 6) \\ &= \frac{1}{10}(5\lambda - 6)(2\lambda + 1) \end{aligned}$$

Thus we have two eigenvalues $\lambda = \frac{6}{5}, \frac{-1}{2}$.

For $\lambda = \frac{6}{5}$, we solve $(A - \frac{6}{5}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{6}{5}I)\mathbf{v} = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{6}{5}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

For $\lambda = \frac{-1}{2}$, we solve $(A - \frac{-1}{2}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{-1}{2}I)\mathbf{v} = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{-1}{2}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -.5 \\ 2 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue λ has $\det(A - \lambda I) = 0$.

The following idea is important in a variety of contexts in this course. For a matrix A, assume we have two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of eigenvalues λ_1, λ_2. Form the matrix

$$M = [\mathbf{v}_1\, \mathbf{v}_2].$$

We have the matrix equation

$$AM = MD$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Now make the assumption that M is invertible. This is a non trivial assumption. For us, it is true as long as $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k. We can verify this to be true if $\lambda_1 \neq \lambda_2$. Assume $\mathbf{v}_1 = k\mathbf{v}_2$ and get a contradiction:

$$A\mathbf{v}_1 = A(k\mathbf{v}_2) = kA(\mathbf{v}_2) = k\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1,$$

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1.$$

We conclude that $\lambda_2\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, i.e. $(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$ and so, with $\mathbf{v}_1 \neq \mathbf{0}$, $\lambda_1 - \lambda_2 = 0$ and so $\lambda_1 = \lambda_2$ which is a contradiction. Thus $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k.

Now

$$AM = MD \text{ means } M^{-1}AM = D \text{ and } A = MDM^{-1}.$$

In our case

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{6}{5} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}$$

There are applications where we wish to know what happens to A^n as $n \longrightarrow \infty$.

An application associated with this matrix is a simple model of a growing bird population. Let

$$x_n = \text{ no. of adults in year } n,$$

$$y_n = \text{ no. of juveniles in year } n.$$

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized in a matrix equation:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

We deduce by induction, that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Now we have $A = MDM^{-1}$ and so

$$A^2 = MDM^{-1}MDM^{-1} = MD(M^{-1}M)DM^{-1} = MD^2M^{-1},$$

$$A^3 = MDM^{-1}MDM^{-1}MDM^{-1} = MD(M^{-1}M)D(M^{-1}M)DM^{-1} = MD^3M^{-1},$$

$$A^n = MD^nM^{-1}.$$

It is straightforward to compute

$$D^n = \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix},$$

hence

$$\begin{aligned} A^n = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}^n &= \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix} \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix} \\ &= \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n & \frac{3}{17}(1.2)^n - \frac{3}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n & \frac{5}{17}(1.2)^n + \frac{12}{17}(-.5)^n \end{bmatrix}. \end{aligned}$$

Thus

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n \end{bmatrix} \approx \begin{bmatrix} \frac{12}{17}(1.2)^n \\ \frac{20}{17}(1.2)^n \end{bmatrix},$$

where we are using the fact that $\lim_{n \to \infty}(-.5)^n = 0$. One aspect of the result is that the population is growing 20% a year and also the ratio of adults to juveniles is approximately $3 : 5$ in a stable population. A ratio sufficiently far from $3 : 5$ would alert the biologist to the likelihood of the population having undergone some environmental disturbance in the recent past.