Sample computation

Let

\[ A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \]

An application associated with this matrix is a simple model of a growing bird population. Let

\[ x_n = \text{no. of adults in year } n, \]

\[ y_n = \text{no. of juveniles in year } n. \]

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized in a matrix equation:

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} =
\begin{bmatrix}
.7 & .3 \\
2 & 0
\end{bmatrix}
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}.
\]

We deduce, by induction, that

\[
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix} = A^n
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}.
\]

This is a sample of many applications where we wish to know what happens to \( A^n \) as \( n \to \infty \).

Recall our computation of eigenvalues/eigenvectors for this matrix:

First we define an eigenvector \( \mathbf{x} \) of eigenvalue \( \lambda \) to be satisfy \( A\mathbf{x} = \lambda \mathbf{x} \) and \( \mathbf{x} \neq \mathbf{0} \). This is equivalent to \( (A - \lambda I)\mathbf{x} = \mathbf{0} \) and \( \mathbf{x} \neq \mathbf{0} \). This can only occur by our previous observations when \( \det(A - \lambda I) = 0 \) and moreover when \( \det(A - \lambda I) = 0 \) we can find an \( \mathbf{x} \neq \mathbf{0} \) with \( A\mathbf{x} = \lambda \mathbf{x} \).

\[
\det(A - \lambda I) = \det\left( \begin{bmatrix} .7 - \lambda & .3 \\ 2 & -\lambda \end{bmatrix} \right)
\]

\[
= ( .7 - \lambda)(-\lambda) - .3 \times 2
\]

\[
= \frac{1}{10}(10\lambda^2 - 7\lambda - 6)
\]

\[
= \frac{1}{10}(5\lambda - 6)(2\lambda + 1)
\]

Thus we have two eigenvalues \( \lambda = \frac{6}{5}, -\frac{1}{2} \).

For \( \lambda = \frac{6}{5} \), we solve \( (A - \frac{6}{5} I)\mathbf{v} = \mathbf{0} \) for \( \mathbf{v} \neq \mathbf{0} \):

\[
(A - \frac{6}{5} I)\mathbf{v} = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The vector \( \mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \) works as an eigenvalue of \( A \) of eigenvalue \( \frac{6}{5} \). We check

\[
\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.
\]
For $\lambda = -\frac{1}{2}$, we solve $(A - \frac{1}{2}I)v = 0$ for $v \neq 0$:

$$(A - \frac{1}{2}I)v = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $v = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ works as an eigenvalue of $A$ of eigenvalue $-\frac{1}{2}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -.5 \\ 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$ 

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue $\lambda$ has $\det(A - \lambda I) = 0$.

The following idea is important in a variety of contexts in this course. For a matrix $A$, assume we have two eigenvectors $v_1, v_2$ of eigenvalues $\lambda_1, \lambda_2$. Form the matrix

$$M = [v_1 \, v_2].$$

We have the matrix equation

$$AM = MD$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$ 

Now make the assumption that $M$ is invertible. This is a non trivial assumption. For us, it is true as long as $v_1 \neq k v_2$ for any $k$. We can verify this to be true if $\lambda_1 \neq \lambda_2$. Assume $v_1 = k v_2$ and get a contradiction:

$$A v_1 = A(k v_2) = k A(v_2) = k \lambda_2 v_2 = \lambda_2 v_1,$$

$$A v_1 = \lambda_1 v_1.$$ 

We conclude that $\lambda_2 v_1 = \lambda_1 v_1$, i.e. $(\lambda_1 - \lambda_2)v_1 = 0$ and so, with $v_1 \neq 0$, $\lambda_1 - \lambda_2 = 0$ and so $\lambda_1 = \lambda_2$ which is a contradiction. Thus $v_1 \neq k v_2$ for any $k$.

Now

$$AM = MD$$

means $M^{-1}AM = D$ and $A = MDM^{-1}$.

In our case

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{1}{17} & \frac{1}{17} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{6}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$ 

Now we have $A = MDM^{-1}$ and so

$$A^2 = MDM^{-1}MDM^{-1} = MD(M^{-1}M)DM^{-1} = MD^2M^{-1},$$

$$A^3 = MDM^{-1}MDM^{-1}MDM^{-1} = MD(M^{-1}M)D(M^{-1}M)DM^{-1} = MD^3M^{-1},$$

$$A^n = MD^nM^{-1}.$$ 

It is straightforward to compute

$$D^n = \begin{bmatrix} \left(\frac{6}{5}\right)^n & 0 \\ 0 & \left(-\frac{1}{2}\right)^n \end{bmatrix}.$$
hence

\[ A^n = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}^n = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} \left(\frac{6}{5}\right)^n & 0 \\ 0 & \left(-\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{3}{17} \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n & \frac{3}{17}(1.2)^n - \frac{3}{17}(-.5)^n \\ \frac{5}{17}(1.2)^n - \frac{3}{17}(-.5)^n & \frac{4}{17}(1.2)^n + \frac{12}{17}(-.5)^n \end{bmatrix}. \]

Thus

\[ A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n \\ \frac{5}{17}(1.2)^n - \frac{3}{17}(-.5)^n \end{bmatrix} \approx \begin{bmatrix} \frac{12}{17}(1.2)^n \\ \frac{5}{17}(1.2)^n \end{bmatrix}, \]

where we are using the fact that \( \lim_{n \to \infty} (-.5)^n = 0 \). One aspect of the result is that the population is growing 20% a year and also the ratio of adults to juveniles is approximately 3 : 5 in a stable population. A ratio sufficiently far from 3 : 5 would alert the biologist to the likelihood of the population having undergone some environmental disturbance in the recent past.