When solving a quadratic (over $\mathbb{R}$) you may find there are no roots but you notice that you get expressions involving $\sqrt{-1}$. Rather than interpret $\sqrt{-1}$ as a number, we can proceed as follows.

We define $$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

and show that with a suitable multiplication, that this is a field. Let $z = a + bi$ and $w = c + di$. Define $$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

This is most naturally interpreted as saying $i^2 = -1$ although the formula could be viewed as an abstract operation. We could interpret elements of $\mathbb{C}$ as 2-tuples and define a multiplication of 2-tuples as $$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$

We then have to verify that this operation combined with the standard addition yields a field. I don’t think that this is so helpful, it is easiest to think of $i = \sqrt{-1}$, but there are always different points of view.

We define the real part of $z$ as $\text{Re}(z) = a$ and the imaginary part of $z$ as $\text{Im}(z) = b$. Here we are using the interpretation $i = \sqrt{-1}$ which we view as imaginary. We say $z \in \mathbb{R}$ when $\text{Im}(z) = 0$ although you might say this is an abuse of notation. We have always done the same with rationals $\mathbb{Q}$ and $\mathbb{R}$ and interpret $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Define in the natural way $$z + w = (a + c) + (b + d)i$$

To check field axioms we need $0 = 0 + 0i$ and $1 = 1 + 0i$ and we need multiplicative inverses

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Thus $z^{-1}$ exists if $z \neq 0$. But doing this computation by hand is some work. $(1/2 + 2i)^{-1} = ?$

One useful operation on complex numbers is complex conjugation. Define $$z = a + bi, \quad \overline{z} = a - bi$$

We note that $z\overline{z} = a^2 + b^2$, which you can see is yielding the multiplicative inverses. Moreover $z\overline{z} \in \mathbb{R}$. You should also note that $z + \overline{z} \in \mathbb{R}$ which we use repeatedly. We can check that $\overline{zw} = \overline{z} \overline{w}$ since

$$\overline{zw} = (ac - bd) - (ad + bc)i \text{ and } \overline{z} \overline{w} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i$$

It is somewhat simpler to note that $\overline{z + w} = \overline{z} + \overline{w}$. Then we obtain very useful formulas. Assume $A\mathbf{x} = \lambda \mathbf{x}$ where we have $A \in \mathbb{C}^{n \times n}$, $\mathbf{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Then

$$\overline{A\mathbf{x}} = \overline{\lambda \mathbf{x}} \text{ becomes } \overline{A\mathbf{x}} = \overline{\lambda} \mathbf{x}$$

If $A \in \mathbb{R}^{n \times n}$, then $\overline{A} = A$. Thus if we have an eigenvector $\mathbf{x}$ of eigenvalue $\lambda$ for $A$, then $\overline{\mathbf{x}}$ is an eigenvector of eigenvalue $\overline{\lambda}$ for $A$.

Example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \det(A - \lambda I) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$
For eigenvalue $\lambda = i$ we find the eigenvector
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-i \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
i \\
\end{bmatrix} \cdot i
\]

Our previous remark gives us an eigenvector of eigenvalue $-i$:
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
i \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
-i \\
\end{bmatrix} \cdot (-i)
\]

This a lovely example of two for the price of one.

Perhaps the most amazing fact is the formula for $e^z = e^{a+bi}$. We note $e^{a+bi} = e^a e^{bi}$. The expression $e^a$ is easy since $a \in \mathbb{R}$. For $e^{bi}$ we try our usual formula for the exponential
\[
e^{bi} = 1 + (bi) + \frac{1}{2!}(bi)^2 + \frac{1}{3!}(bi)^3 + \frac{1}{4!}(bi)^4 + \frac{1}{5!}(bi)^5 + \frac{1}{6!}(bi)^6 + \cdots
\]
\[
= 1 - \frac{1}{2!}(b)^2 + \frac{1}{4!}(b)^4 - \frac{1}{6!}(b)^6 + \cdots
\]
\[
+ i \left(b - \frac{1}{3!}(b)^3 + \frac{1}{5!}(b)^5 + \cdots\right)
\]
\[
= \cos b + (\sin b)i.
\]

This is amazing in that it relates the exponential function to the sine and cosine functions which may come as quite a surprise. They are not usually spoken of together in your earlier courses.

A DE system that relates to this is the following
\[
\frac{d}{dt} y(t) = -y(t)
\]

This is a second order DE. But by introducing the derivative $y'(t) = \frac{d}{dt} y(t)$ we have
\[
\frac{d}{dt} \begin{bmatrix}
y(t) \\
y'(t) \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0 \\
\end{bmatrix} \begin{bmatrix}
y(t) \\
y'(t) \\
\end{bmatrix}
\]

In analogy to our previous solutions of DE’s, we obtain a general solution
\[
\begin{bmatrix}
y(t) \\
y'(t) \\
\end{bmatrix} = c_1 e^{it} \begin{bmatrix}
-i \\
1 \\
\end{bmatrix} + c_2 e^{-it} \begin{bmatrix}
i \\
1 \\
\end{bmatrix} = c_1 (\cos(t) + i \sin(t)) \begin{bmatrix}
-i \\
1 \\
\end{bmatrix} + c_2 (\cos(t) - i \sin(t)) \begin{bmatrix}
i \\
1 \\
\end{bmatrix}
\]

You should be slightly worried that we are writing what looks like complex functions for a problem which is surely restricted to reals. If we start with the initial conditions $y(0) = 1$ and $y'(0) = 0$ we can solve for $c_1, c_2$ and hopefully real solutions result.
\[
\begin{bmatrix}
y(0) \\
y'(0) \\
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
\end{bmatrix} = c_1 \begin{bmatrix}
-i \\
1 \\
\end{bmatrix} + c_2 \begin{bmatrix}
i \\
1 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-i \\
i \\
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
\end{bmatrix}
\]

So
\[
\begin{bmatrix}
c_1 \\
c_2 \\
\end{bmatrix} = \begin{bmatrix}
-i \\
i \\
\end{bmatrix}^{-1} \begin{bmatrix}
1 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{i}{2} \\
\frac{1}{2} \\
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} \\
\frac{1}{2} \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2}i \\
\frac{1}{2}i \\
\end{bmatrix}
\]
We compute
\[
\begin{bmatrix}
y(t)
y'(t)
\end{bmatrix} = \frac{1}{2} i \cdot (\cos(t) + i \sin(t)) \begin{bmatrix}
-i
1
\end{bmatrix} + \frac{1}{2} i \cdot (\cos(t) - i \sin(t)) \begin{bmatrix}
i
1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\frac{1}{2}(\cos(t) + i \sin(t)) + \frac{1}{2}(\cos(t) - i \sin(t)) \\
\frac{1}{2}(\sin(t) + i \cos(t)) + \frac{1}{2}(- \sin(t) - i \cos(t))
\end{bmatrix} = \begin{bmatrix}
\cos(t) \\
-\sin(t)
\end{bmatrix}
\]
which is easily checked as the solution to the differential equations and satisfies the initial conditions.

You may note that \(c_1 = c_2, \ e^{it} = e^{-it}\), and the eigenvectors are conjugates and so the result must be real!