First consider the system of DE’s which we motivated in class using water passing through two tanks while flushing out salt contamination.

\[
\begin{align*}
y_1'(t) &= -\frac{1}{10} y_1(t) + \frac{1}{40} y_2(t), \\
y_2'(t) &= \frac{1}{10} y_1(t) - \frac{1}{10} y_2(t), \\
y_1(0) &= 60, \\n_2(0) &= 0
\end{align*}
\]

\[\frac{d}{dt} y = Ay\] where \(A = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}\)

We may compute

\[\begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -3/20 & 0 \\ 0 & -1/20 \end{bmatrix} \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix} = MDM^{-1}\]

I offer several solutions, the first using change of variable (change of basis). The second considers the matrix exponential \((y = e^{At}y(0))\) and the third solution considers write the final solution as a linear combination of solutions to the DE to satisfy the initial conditions.

Our first idea was to rewrite \(\frac{d}{dt} y = Ay\) as \(\frac{d}{dt} y = MDM^{-1}y\) and then \(M^{-1} \frac{d}{dt} y = DM^{-1}y\). Then using the linearity of differentiation, we have \(\frac{d}{dt} M^{-1}y = DM^{-1}y\). We set

\[z = M^{-1}y\]

and obtain the ‘easy’ system of differential equations

\[\frac{d}{dt} z = Dz\]

namely

\[\frac{d}{dt} z_1(t) = (-3/20) z_1(t),\]
\[\frac{d}{dt} z_2(t) = (-1/20) z_2(t),\]

which we solve as

\[z_1(t) = z_1(0)e^{(-3/20)t}, \quad z_2(t) = z_2(0)e^{(-1/20)t}.\]

We may compute \(z_1(0), z_2(0)\) using \(y_1(0) = 60\) and \(y_2(0) = 0\) and \(z = M^{-1}y\) to obtain \(z_1(0) = 30\) and \(z_2(0) = 30\). Now we use \(y = Mz\) and obtain

\[\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 30e^{(-3/20)t} \\ 30e^{(-1/20)t} \end{bmatrix} = \begin{bmatrix} 30e^{(-3/20)t} + 30e^{(-1/20)t} \\ -60e^{(-1/20)t} + 60e^{(-1/20)t} \end{bmatrix}\]

This solution technique of changing variables (changing basis) to make the system of differential equations easy to solve (diagonalization) follows our usual pattern.

A second solution involves tackling the problem directly in what first appears a bit unlikely:

\[\frac{d}{dt} y = Ay, \quad y = e^{At}y(0)\]

Recall that

\[e^{At} = I + At + \frac{1}{2!} A^2t^2 + \frac{1}{3!} A^3t^3 + \ldots\]
\[
\frac{d}{dt} e^{At} = 0 + A + A^2t + \frac{1}{2} A^3t^2 + \cdots = Ae^{At}
\]
where the derivative has been done entrywise. We note that \(e^{At}\) at \(t = 0\) is in fact \(e^0 = I\) (we are exponentiating the \(2 \times 2\) zero matrix) and hence \(e^{At}y(0)\) at \(t = 0\) is indeed \(y(0)\) as desired.

We have techniques for doing this namely:
\[
y = e^{At}y(0) = Me^{Dt}M^{-1}y(0)
\]
This technique can be used for non diagonalizable matrices \(A\) if we can find a similar matrix \(B\) (i.e. \(A = MBM^{-1}\)) for which \(e^{Bt}\) is easy to compute.

A third solution technique is commonly used when solving DE's, we write our solution in vector form in terms of the eigenvectors
\[
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 30e^{(-3/20)t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 30e^{(-1/20)t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
This suggests another solution strategy. We seek solutions of the form \(e^{At}v\) with \(\frac{d}{dt} e^{At}v = \lambda e^{At}v = e^{At}Av\) by solving \(Av = \lambda v\) and hence solving for eigenvalues and eigenvectors of \(A\). Then (it needs to be proven!) we write an arbitrary solution to the system of DE's as
\[
y = ae^{(-3/20)t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + be^{(-1/20)t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
and solving for \(a, b\) given the values \(y(0)\).

Consider the system
\[
\begin{align*}
y_1'(t) &= + y_2(t) \\
y_2'(t) &= -4y_1(t) + 4y_2(t)
\end{align*}
\]
y_1(0) = 60, y_2(0) = 0

We begin by noticing the following (how to obtain this is not part of your course this term)
\[
A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}
\]
\[
\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}
\]
We can solve the system of DE's by the matrix exponential idea: \(y(t) = e^{At}y(0) = Me^{St}M^{-1}y(0)\)

We now consider a system of DE's that has complex eigenvalues. It arises from considering the Differential Equation
\[
y'' = -y, \quad y(0) = 1, y'(0) = 0
\]
If we set \(y_1(t) = y\) and \(y_2(t) = y'\) then we can set
\[
y = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\]
and then we can write the DE in vector form as
\[
\frac{d}{dt}y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}y
\]
We can compute eigenvalues and eigenvectors in the natural way using $\mathbf{C}$ instead of $\mathbf{R}$.

$$
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
i & -i \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{2}i & -\frac{1}{2} \\
\frac{1}{2}i & -\frac{1}{2}
\end{bmatrix}
= 
\begin{bmatrix}
i & -i \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
i & 0 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{2}i & -\frac{1}{2} \\
\frac{1}{2}i & -\frac{1}{2}
\end{bmatrix}
A M^{-1}
$$

We could use either of the three methods from above. We can use our third method above (that follows from our change of basis idea). Let $\mathbf{v}_i$ be an eigenvector of eigenvalue $\lambda_i$. Then as solution to the DE, ignoring initial conditions, is

$$y = e^{\lambda_i} \mathbf{v}_i$$

In order to match the initial conditions, we take the appropriate linear combination of these solutions from eigenvector/eigenvalue pairs. In our case we have

$$
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix}
= ae^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}
+ be^{-it} \begin{bmatrix} -i \\ -1 \end{bmatrix}
$$

We can solve for $a, b$ by setting $t = 0$, noting $e^0 = 1$, to obtain

$$
\begin{bmatrix}
y_1(0) \\
y_2(0)
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
= a \begin{bmatrix} i \\ -1 \end{bmatrix}
+ b \begin{bmatrix} -i \\ -1 \end{bmatrix}
= M \begin{bmatrix} a \\ b \end{bmatrix}
$$

We then solve for $a, b$ using $M^{-1}$ to obtain

$$
\begin{bmatrix} a \\ b \end{bmatrix}
= M^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
= \begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
= \begin{bmatrix} -\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}
$$

We then can compute the solution.

Once, in a previous version of 223, I solved this by substituting

$$e^{it} = \cos(t) + i \sin(t), \quad e^{-it} = \cos(-t) + i \sin(-t) = \cos(t) - i \sin(t)$$

Then I proceeded to solve for $a, b$ which made things much more complicated. Setting $t = 0$ first and then solving for $a, b$ makes things easier. This is easier for computations; both methods spit out an answer. The solution becomes

$$y = -\frac{1}{2}i(\cos(t) + i \sin(t)) \begin{bmatrix} i \\ -1 \end{bmatrix}
+ \frac{1}{2}i(\cos(t) - i \sin(t)) \begin{bmatrix} -i \\ -1 \end{bmatrix}
= \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

Thus the solution to our DE as expected is $y = \cos(t)$ which has $y(0) = 1$ and $y'(0) = 0$.

We can make some additional simplifications to save work. Let $z = c + di \in \mathbf{C}$. Use the notation $Re(z) = c$ and $Im(z) = d$ to denote the real and imaginary part of $z$ although I would caution that $Im(z) \in \mathbf{R}$. In addition this conflicts with our definition $Im(f)$ referring to the image of the function $f$. Sigh. We note that $z + \bar{z} \in \mathbf{R}$. Since we are going to get a real solution we can deduce that in the expression

$$
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix}
= a_1 e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}
+ a_2 e^{-it} \begin{bmatrix} -i \\ -1 \end{bmatrix}
$$

that $a_1 = a_2$. We can get two different real solutions from the Real and Imaginary parts of one solution

$$
\begin{bmatrix} i \\ -1 \end{bmatrix}
= \begin{bmatrix} \cos t + i \sin t \\ -\sin t \end{bmatrix}
+ \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}
$$
\[ \text{Re}(e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}) = \text{Re}((\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} \]

\[ \text{Im}(e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}) = \text{Im}((\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \]

You may verify that the real part comes from the choice \( a_1 = 1/2, a_2 = 1/2 \) and the imaginary part comes from the choice \( a_1 = -i/2, a_2 = i/2 \). We now solve taking a linear combination of these two solutions (which are both real although their origin was complex):

\[
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + b \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

We solve and get \( a = 0, b = 1 \) yielding the solution \( y_1(t) = \cos t, y_2(t) = -\sin t \).

It is not particularly helpful to note that we can compute \( e^{At} \) for this \( A \) without using complex numbers. For this problem

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\]

from which we have expressions for all powers of \( A \). Then

\[
e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \frac{1}{5!} \begin{bmatrix} 0 & t^5 \\ -t^5 & 0 \end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix} 1 + 0 - \frac{1}{2!} t^2 + 0 + \frac{1}{3!} t^4 + 0 \cdots \\ 0 + t + 0 - \frac{1}{3!} t^3 + 0 + \frac{1}{4!} t^5 \cdots \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}
\]