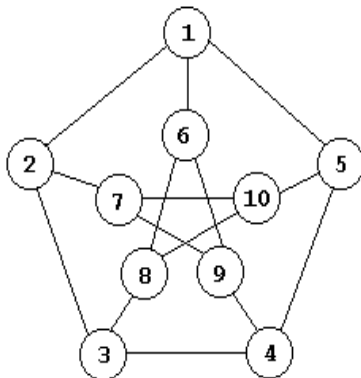


## MATH 223: A Graph Decomposition result.

The Petersen graph is a well known graph that often appears as an example or counterexample of graph properties. Given that it is a graph on 10 vertices with each vertex being adjacent to 3 edges, it is possible to ask the question whether the complete graph on 10 vertices can be decomposed into 3 copies of the Petersen graph.



Let  $A$  denote the adjacency matrix of the Petersen graph.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Let  $J$  denote the  $10 \times 10$  matrix of all 1's. Then  $J - I$  is the adjacency matrix of the complete graph on 10 vertices.

The matrix  $A$  satisfies the remarkable matrix equation

$$A^2 + A = 2I + J.$$

This follows from noticing that any pair of vertices not already joined by an edge are joined by a unique path of length 2; namely there is a unique vertex which is joined to both.

From this equation or directly, we can deduce that  $\det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4$  and that  $\mathbf{1}$  is an eigenvector of eigenvalue 3. How could one approach this computation without just computing the  $10 \times 10$  determinant? Note that  $J$  has eigenvalue 10 (of multiplicity 1 with eigenvector  $\mathbf{1}$ ) and eigenvalue 0 of multiplicity 9. Thus  $2I + J$  has eigenvalue 12 of multiplicity 1 and eigenvalue 2 of multiplicity 9. Now if  $\mathbf{x}$  is an eigenvector of  $A$  of eigenvalue  $\lambda$  then  $\mathbf{x}$  is an eigenvector of  $A^2 + A$  of eigenvalue  $\lambda^2 + \lambda$ . Thus the possible eigenvalues for  $A$  are prescribed. We already know that  $\mathbf{1}$  is an eigenvector of  $A$  of eigenvalue 3 (or at least it is easy to check). The other 9 eigenvalues must satisfy  $\lambda^2 + \lambda = 2$  thus either 1 or  $-2$  with total multiplicity being 9. Now the trace of  $A$ , which is 0, is the sum of the eigenvalues and so we deduce that 1 has multiplicity 5 and  $-2$  has multiplicity 4.

The decomposition we are looking for takes the matrix form

$$J - I = A_1 + A_2 + A_3$$

where  $A_1, A_2, A_3$  are each obtained from  $A$  by a simultaneous row and column permutation, i.e.  $A_i = PAP^T = PAP^{-1}$  where  $P$  is a permutation matrix. Each  $A_i$  is similar to  $A$ . Also,  $A_i$  is a symmetric matrix and hence the dimension of an eigenspace of eigenvalue  $\lambda$  is the multiplicity of  $\lambda$  as a root in the characteristic equation for  $A_i$ . We note that

$$\det(A_i - \lambda I) = \det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4.$$

For all three matrices  $A_1, A_2, A_3$ , we note that  $\mathbf{1}$  is an eigenvector of eigenvalue 3 and so each has a 5-dimensional eigenspace of eigenvalue 1 orthogonal to  $\mathbf{1}$  and hence contained in  $(\text{span}(\mathbf{1}))^\perp$  which is a 9-dimensional space. Now two 5-dimensional subspaces of a 9-dimensional space must have a non zero vector, say  $\mathbf{v}$ , in their intersection. Thus there is a vector  $\mathbf{v}$  that is an eigenvector of eigenvalue 1 for both  $A_1$  and  $A_2$  ( $A_1\mathbf{v} = A_2\mathbf{v} = \mathbf{v}$ ) and moreover  $\mathbf{v}$  is orthogonal to  $\mathbf{1}$ . We compute (using  $\mathbf{1} \cdot \mathbf{v} = 0$ )

$$(J - I)\mathbf{v} = J\mathbf{v} - \mathbf{v} = -\mathbf{v},$$

$$(A_1 + A_2 + A_3)\mathbf{v} = A_1\mathbf{v} + A_2\mathbf{v} + A_3\mathbf{v} = 2\mathbf{v} + A_3\mathbf{v}$$

We deduce that

$$A_3\mathbf{v} = -3\mathbf{v}.$$

This is a contradiction since  $-3$  is not an eigenvalue of  $A_3$ .

Thus the desired decomposition cannot exist. This result can be modestly generalized but I introduced it because the problem reviews so many facts from our MATH 223 course.

A different direction to take with the Petersen Graph is to show that for any pair of vertices they are either joined by an edge or a path of length 2 using the fact that there are three eigenvalues for the diagonalizable adjacency matrix  $A$ . In particular we can readily show using diagonalizability that  $(A - 3I)(A - I)(A + 2I)$  which gives an expression for  $A^3$  in terms of  $I, A, A^2$ . Thus

$$\text{span}\{I, A, A^2, A^3, A^4, \dots\} = \text{span}\{I, A, A^2\}. \quad \dim(\text{span}\{I, A, A^2, A^3, A^4, \dots\}) = 3.$$

Now we consider powers of  $A$  from a different point of view. In particular the  $i, j$  entry of  $A^k$  counts the number of walks from  $i$  to  $j$  of  $k$  edges. This can be proven by induction if you wish. Thus if we have a pair of vertices  $i, j$  joined by a walk of three edges and not one edge or two edges, then the  $i, j$  entry of  $A^3$  is nonzero while the  $i, j$  entries of  $I, A, A^2$  are zero. Thus

$$A^3 \notin \text{span}\{I, A, A^2\}$$

a contradiction. Thus we conclude that every pair of vertices are joined by an edge or a walk of two edges simply from the fact that  $A$  had only 3 eigenvalues. The reverse implication will not work.