You must show your work and explain your answers. You may certainly use results given in class. If you use theorems not given in class, be kind enough to state them.

1. (20 marks) 

\[
A = \begin{bmatrix}
1 & 1 & 2 & 1 & -1 & 1 \\
2 & 2 & 5 & 4 & -2 & 4 \\
2 & 2 & 6 & 6 & -2 & 6 \\
1 & 1 & 3 & 3 & 0 & 3
\end{bmatrix}
\]

There is an invertible matrix \( B \) so that 

\[
BA = \begin{bmatrix}
1 & 1 & 2 & 1 & -1 & 1 \\
0 & 0 & 1 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

a) [3 marks] \( \text{rank}(A) = 3 \) (number of pivots in \( BA \))

b) [8 marks]

\[
\text{basis for the row space} \begin{bmatrix}
1 & 1 & 2 & 1 & -1 & 1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
1 & 2 & 0 & 0
\end{bmatrix}
\]

\[
\text{basis for the column space} \begin{bmatrix}
1 & 2 & -1 \\
2 & 5 & -2 \\
2 & 6 & -2 \\
1 & 3 & 0
\end{bmatrix}
\]

The matrix \( A \) was chosen so that the first 3 rows of \( A \) were not a basis for the row space of \( A \).

c) [5 marks]

\[
\text{basis for the null space} \begin{bmatrix}
3 & 3 & -1 \\
0 & 0 & 1 \\
-2 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

d) [4 marks] Consider a vector \( c \in \mathbb{R}^4 \) so that \( Ax = c \) is inconsistent (i.e. the system of equations has no solution). Let \( A' = [c | A] \), i.e. \( A' \) is the \( 4 \times 7 \) matrix with \( c \) being the first column and the remainder being the columns of \( A \). What is \( \text{rank}(A') \)? We note that any vector \( Ax \) is in the column space of \( A \). We are told that \( Ax = c \) is inconsistent and so \( c \notin \text{column space}(A) \). Thus \( \text{column space}(A') \) has dimension one larger than \( A \), and so \( \text{rank}(A') = 4 \).

2. (20 marks) Consider the system of differential equations

\[
\frac{d}{dt} x_1(t) = 5x_1(t) - 3x_2(t) \\
\frac{d}{dt} x_2(t) = 6x_1(t) - 4x_2(t)
\]
You may find it useful to note that:

\[
\begin{bmatrix}
5 & -3 \\
6 & -4 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
1 & 2 \\
\end{bmatrix} \begin{bmatrix}
2 & 0 \\
0 & -1 \\
\end{bmatrix} \begin{bmatrix}
2 & -1 \\
-1 & 1 \\
\end{bmatrix}
\]

a) (15 marks) Find the general solution for \(x_1(t), x_2(t)\) as a function of \(x_1(0), x_2(0)\).

\[
x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

We solve

\[
\begin{bmatrix}
x_1(0) \\
x_2(0) \\
\end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\
1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

and so

\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\
-1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2x_1(0) - x_2(0) \\
x_1(0) + x_2(0) \end{bmatrix}
\]

and so general solution is

\[
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\end{bmatrix} = (2x_1(0) - x_2(0)) e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (x_1(0) + x_2(0)) e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

b) (5 marks) Given \(x_1(0) = 1\) and \(x_2(0) = 2\), then \(2x_1(0) - x_2(0) = 0\) and \(-x_1(0) + x_2(0) = 1\). Then \(x_1(t) = e^{-t}\) and \(x_2(t) = 2e^{-t}\). We compute

\[
\lim_{t \to \infty} \frac{x_1(t)}{x_2(t)} = \lim_{t \to \infty} \frac{e^{-t}}{2e^{-t}} = \frac{1}{2}.
\]

I had originally chosen values that did not have \(c_1 = 0\) and then the limit is 1.

3. (20 marks)

Let \(u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}, \ u_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\)

\[
\text{NOTE: } \begin{bmatrix} 1 & 0 & 0 \\
1 & -3 & 2 \\
1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\
-1 & -1 & 2 \\
-2 & -1 & 3 \end{bmatrix}
\]

Let \(f : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) be the linear transformation satisfying

\[
f(u_1) = u_2 + u_3, \quad f(u_2) = 2u_3, \quad f(u_3) = 2u_2.
\]

a) (5 marks) Show that \(u_2 + u_3\) is an eigenvector for \(f\).

\[
f(u_2 + u_3) = f(u_2) + f(u_3) = 2u_3 + 2u_2 = 2(u_2 + u_3)\]

and so \(u_2 + u_3\) is an eigenvector of eigenvalue 2.

b) (5 marks) Give the matrix representation of \(f\) with respect to the basis \(\{u_1, u_2, u_3\}\).

\[
\begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 2 \\
1 & 2 & 0 \end{bmatrix}
\]

\(f\) wrt \(U\)
c) (10 marks) Give the matrix representation of $f$ with respect to the basis $\{e_1, e_2, e_3\}$ (the standard basis). Give the explicit matrix with integer entries. Also give an eigenvalue of the matrix.

$$
\begin{bmatrix}
1 & 0 & 0 \\
1 & -3 & 2 \\
1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
E \leftarrow U \\
T \text{ wrt } U \\
U \leftarrow E
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 2 \\
1 & 2 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
-1 & -1 & 2 \\
-2 & -1 & 3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 \\
7 & 2 & -10 \\
2 & 0 & -2
\end{bmatrix}
$$

We note from part a), that 2 will be an eigenvalue (being an eigenvalue is a property of the linear transformation).

4. (6 marks) Let

$$B = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.$$

If $A$ is similar to $B$, what is $\dim(\text{nullspace}(A))$?

We note that $\text{rank}(B) = 2$ (it is already in staircase pattern with 2 pivots). Now assume $AM = MB$ where $M$ is invertible. Then row space($B$) = row space($MB$) ($M$ is invertible) and so $\text{rank}(MB) = \text{rank}(B) = 2$. But then since $AM = MB$, we have $\text{rank}(AM) = 2$. Now since $M$ is invertible, column space($A$) = column space($AM$) and so $\text{rank}(A) = \text{rank}(AM) = 2$. But now $\dim(\text{nullspace}(A)) = 3 - 2 = 1$.

5. (7 marks) Let $A$ be a $3 \times 3$ diagonalizable matrix with eigenvalues -1,1,4. What is $\det(A + A^{-1})$?

We can use the result from the first midterm that $x$ is an eigenvector of $A$ of eigenvalue $\mu$ if and only if $x$ is an eigenvector of $A^{-1}$ of eigenvalue $1/\mu$. Thus $A^{-1}$ has eigenvalues $1/(-1), 1/1, 1/2$ with the corresponding eigenvectors and hence the same diagonalizing matrix $M$. Thus

$$(A + A^{-1}) = M\left(\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{bmatrix} + \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/4
\end{bmatrix}\right)M^{-1} = M\left(\begin{bmatrix}
-2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 17/4
\end{bmatrix}\right)M^{-1}$$

Using the product rule,

$$\det(A + A^{-1}) = \det\left(\begin{bmatrix}
-2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 5/2
\end{bmatrix}\right) = (-2) \times 2 \times 5/2 = -17$$

6. (7 marks) Given a square matrix $A$, explain why $\text{rank}(A + A^2) \leq \text{rank}(A)$.

We see that column space($A + A^2$) = column space($A(I + A)$) $\subseteq$ column space($A$), a result from class that multiplying on the right does column operations and so the resulting columns of $AB$ are in the column space of $A$ for any $B$. Then $\text{rank}(A + A^2) = \dim(\text{column space}(A + A^2))$ and $\text{rank}(A) = \dim(\text{column space}(A))$ and the containment column space($A + A^2$) = column space($A(I + A)$) $\subseteq$ column space($A$) gives the inequality.

Alternatively we can verify that since columns of $A^2$ are linear combinations of columns of $A$, then column space($A^2$) $\subseteq$ column space($A$). Then column space($A + A^2$) $\subseteq$ column space($A$).

Alternatively we can verify that if $x \in \text{nullspace}(A)$, then $(A + A^2)x = Ax + AAx = 0 + 0 = 0$ and so $x \in \text{nullspace}(A + A^2)$. Thus $\text{nullspace}(A) \subseteq \text{nullspace}(A + A^2)$ and so $\dim(\text{nullspace}(A)) \leq \dim(\text{nullspace}(A + A^2))$ which yields the result.
7. (10 marks) Let \(A\) be a \(4 \times 4\) matrix. Assume that the only eigenvalues for \(A\) are 1,2. Assume \(\dim(\text{nullspace}(A - 2I)) = 2\). What are the possibilities for the characteristic polynomial \(\det(A - \lambda I)\)?

You can express a degree 4 polynomial as a product of four linear factors (if it factors into linear factors) with no need to expand into the individual terms.

We have that \(\text{nullspace}(A - 2I)\) is the eigenspace for eigenvalue 2. We use the result from an assignment that the dimension of an eigenspace for eigenvalue \(k\) is a lower bound on the multiplicity of \(k\) as a root in the characteristic polynomial. Thus \((\lambda - 2)^2\) divides the characteristic polynomial. Since 1 is also an eigenvalue, then \((\lambda - 1)\) also divides the characteristic polynomial. The characteristic polynomial is a quartic polynomial and is divisible by \((\lambda - 2)^2(\lambda - 1)\) so it is a product of four linear factors. The remaining linear factor must be \((\lambda - 1)\) or \((\lambda - 2)\) in view of the fact that the only eigenvalues are 1,2.

Thus the characteristic polynomial is either \((\lambda - 2)^2(\lambda - 1)^2\) or \((\lambda - 2)^3(\lambda - 1)\) and both are possible.

8. (10 marks) Let \(M_3\times 3\) denote the vector space of \(3 \times 3\) matrices and let \(A, B\) be given matrices in \(M_3\times 3\).

Let 0 denote the zero matrix. Define the following subset of \(M_3\times 3\).

\[ V = \{ X \in M_3\times 3 : AXB = 0 \} \]

a) (4 marks) Show that \(V\) is a vector space. We need only verify closure. Assume \(X, Y \in V\) so that \(AXB = AYB = 0\). We check \(A(X + Y)B = (AX + AY)B = AXY + AYB = 0 + 0 = 0\) and so \(X + Y \in V\). Also \(A(kX)B = k(AXB) = k \cdot 0 = 0\) and so \(kX \in V\). And finally \(A0B = 0B = 0\) and so \(0 \in V\). (this last check is needed to verify \(V\) is nonempty.

b) (2 marks) If \(A, B\) are both invertible, what is \(\dim(V)\)? Assume \(A^{-1}, B^{-1}\) exist. Then the equation \(AXB = 0\) can be rewritten as \(A^{-1}AXB^{-1} = A^{-1}0B^{-1}\) which yields \(X = 0\). Thus \(V = \{0\}\) and so \(\dim(V) = 0\).

c) (4 marks) If \(B\) is invertible and rank\((A) = 2\), what is \(\dim(V)\)? Again we assume \(B^{-1}\) exists and then the equation \(AXB = 0\) can be rewritten as \(AXBBB^{-1} = 0B^{-1}\) which yields \(AX = 0\). Thus each column of \(X\) is in nullspace\((A)\). Now rank\((A) = 2\) and rank\((A) + \dim(\text{nullsp}(A)) = 3\) so \(\dim(\text{nullspace}(A)) = 1\). Let \(\{u\}\) be a basis for \(\dim(\text{nullspace}(A))\). Then each column of \(X\) is a multiple of \(u\). Then a basis for \(V\) is

\[ \{ [u \ 0 \ 0], [0 \ u \ 0], [0 \ 0 \ u] \} \]

and so \(\dim(V) = 3\).