1. Determine bases for the following subspaces of $\mathbb{R}^3$.
   a) the line $x = 5t, y = -2t, z = t$.
   b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$.

2. Let
   \[
   A = \begin{bmatrix}
   0 & 1 & 1 & 2 & -3 & 1 \\
   0 & 2 & 0 & 6 & -6 & 0 \\
   0 & 3 & 7 & 2 & -9 & 7 \\
   0 & 2 & 2 & 4 & -4 & 3
   \end{bmatrix}
   \]
   Determine a basis for the column space of $A$ (chosen from columns of $A$) and determine a basis for the row space of $A$. Also give a basis for the nullspace of $A$, namely $\{x \in \mathbb{R}^6 : Ax = 0\}$.

   Such questions appear on your practice for Midterm 2 although in those questions the reduction to staircase pattern has already occurred, saving you some computation.

3. Show that the set of all vectors $(b_1, b_2, b_3, b_4)^T$ such that the system below is consistent (i.e. can be solved)
   \[
   \begin{bmatrix}
   2 & 3 & 1 \\
   4 & 3 & 3 \\
   1 & 3 & 0 \\
   2 & 0 & 2
   \end{bmatrix}
   \begin{bmatrix}
   x
   \end{bmatrix}
   =
   \begin{bmatrix}
   b_1 \\
   b_2 \\
   b_3 \\
   b_4
   \end{bmatrix}
   \]
   is a subspace of $\mathbb{R}^4$. Then find a basis of the subspace.

4. We say two $n \times n$ matrices $A, B$ are similar if there is an invertible matrix $M$ with $A = MBM^{-1}$. We have shown (assignment 4, question 5) that $A, B$ being similar implies $\det(A - \lambda I) = \det(B - \lambda I)$. Assume $A$ has $k \leq n$ linearly independent eigenvectors $u_1, u_2, \ldots, u_k$ all of eigenvalue 2.
   a) Explain why we can always extend $u_1, u_2, \ldots, u_k$ to an invertible $n \times n$ matrix $M$ where the first $k$ columns of $M$ are $u_1, u_2, \ldots, u_k$. 
   b) Assume that the dimension of the eigenspace of $A$ of eigenvalue 2 is exactly $k$. Show that the multiplicity of 2 as a root of $\det(A - \lambda I)$ is at least $k$. Hint: Extend the result of assignment 4 question 4 by using the $k$ linearly independent eigenvectors $u_1, u_2, \ldots, u_k$ of $A$ of eigenvalue 2 in $M$ and then obtaining the first $k$ columns of the matrix $B$ where $A = MBM^{-1}$.
   Comment and not a hint: If $A$ were diagonalizable with $A = MDM^{-1}$ for a diagonal matrix $D$, then we know automatically that $\dim(\text{eigenspace of } A \text{ for } \lambda = 2) \geq k$ as a root in $\det(A - \lambda I) = \det(D - \lambda I)$ and so we have equality! There are other ways of seeing this for diagonalizable matrices. Our result of this question seems helpful for non-diagonalizable matrices.

5. Let $A$ be an $n \times n$ matrix with various eigenvalues including $\lambda$ and $\mu$ with $\lambda \neq \mu$. Let $L, M$ be the eigenspaces associated with eigenvalues $\lambda, \mu$ respectively. Let $\{u_1, u_2, \ldots, u_p\}$ be a basis for $L$ and let $\{v_1, v_2, \ldots, v_q\}$ be a basis for $M$. Show that $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}$ is a linearly independent set of $p + q$ vectors. (Hint: try $p = 1$ and $q = 1$ to start). Comment: You could explore the case if there were three different eigenvalues and three bases for the eigenspaces).

6. Let $M_{3 \times 3}$ denote the vector space of all $3 \times 3$ matrices (over $\mathbb{R}$). Consider following transformation
   \[f : M_{3 \times 3} \to M_{3 \times 3}, f(A) = A^T\]
   Show that this is a linear transformation.

   We say that a matrix $A$ is symmetric if $A^T = A$ and we say that a matrix $A$ is skew-symmetric if $A^T = -A$. 

MATH 223 Assignment #6 due Friday October 28.
a) Warmup question: Give a basis for $M_{3\times3}$.

b) What is the dimension of the eigenspace of eigenvalue 1 for $f$? Explain.

c) What is the dimension of the eigenspace of eigenvalue -1 for $f$? Explain.

d) Now use the previous question (and other facts) to show that any $A \in M_{3\times3}$ is a linear combination of a symmetric matrix and a skew-symmetric matrix (you could show this directly of course but I’m asking you to use linear independence/dimension arguments).

7. Write a proof of the result quoted in class namely that for every matrix $A$, the rank($A$) is the maximum $k$ such that $A$ has a $k \times k$ submatrix which is invertible. Recall that a submatrix of $A$ is obtained by deleting rows and columns (such a minor in our definition of determinant) or vice versa by selecting the matrix whose entries are in specified rows and columns of $A$. The set of rows and columns selected can be quite different.

8. Let $A$ be an $n \times n$ matrix. Assume that we have factored det($A - \lambda I$) into linear factors, namely
\[ \det(A - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda). \]
Show that
\[ \prod_{i=1}^{n} \lambda_i = \det(A) \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = \text{tr}(A). \]

The second is less obvious but consider the coefficient of $\lambda^{n-1}$ in $\det(A - \lambda I)$. You may use this result in other work even if you can’t prove it here.