1. Given the three vectors
\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]
consider each of the 6 possible orderings of the vectors as three columns of a 3 \( \times \) 3 matrix. The three orderings corresponding to the right hand rule are the rotations of \( e_1, e_2, e_3 \) so the orderings \( e_1, e_2, e_3 \) and \( e_2, e_3, e_1 \) and \( e_3, e_1, e_2 \) which correspond to the following three matrices
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and we readily verify that they have determinant +1. The remaining three orderings \( e_1, e_3, e_2 \) and \( e_2, e_1, e_3 \) and \( e_3, e_2, e_1 \) which correspond to the following three matrices
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and we readily verify that they have determinant -1.

2. Which of the following are subspaces of \( \mathbb{R}^3 \)?
   a) We see that \( V = \{(a, 0, 0)^T : a \in \mathbb{R}\} \subseteq \mathbb{R}^3 \). We verify that \( (a, 0, 0)^T + (b, 0, 0)^T = (a+b, 0, 0)^T \in V \) and \( k(a, 0, 0)^T = (ka, 0, 0)^T \in V \) and so \( V \) is a vector space. Alternatively we could verify that \( \text{Vspan}\{(1, 0, 0)^T\} \). Any span is a vector space. This idea can be used in other questions but checking closure is typically easier than finding a spanning set.
   b) We have \( V = \{(a, b, c)^T : c = a + b, \quad a, b, c \in \mathbb{R}\} \subseteq \mathbb{R}^3 \). We verify that \( (a, b, c)^T + (d, e, f)^T = (a+d, b+e, c+f)^T \) and if \( a + b = c \) and \( d + e = f \), then \( a + b + d + e = (a+d) + (b+e) = c + f \). Thus \( (a + d, b + e, c + f)^T \in V \). Similarly, \( k(a, b, c)^T = (ka, kb, kc)^T \) and if \( a + b = c \) then \( k(a + b) = ka + kb = kc \). Thus \( (ka, kb, kc)^T \in V \). Thus, by closure, \( V \) is a vector space. Again we have \( V = \text{span}\{(1, 0, 1)^T, (0, 1, 1)^T\} \) but this requires more careful checking.

3. Which of the following are subspaces of the vector space of all functions \( f \) with domain \( \mathbb{R} \) and range contained in \( \mathbb{R} \)?
   a) Let \( V = \{f : f(-1) = 0 \text{ for all } x \in \mathbb{R}\} \). We see that for \( f, g \in V \) we have \( (f + g)(-1) = f(-1) + g(-1) = 0 + 0 = 0 \) (so \( f + g \in V \)) and \( (kf)(-1) = k \cdot f(-1) = k \cdot 0 = 0 \). Hence \( kf \in V \). Thus \( V \) is a vector space.
   b) Let \( V = \{f : f(x) \leq 0 \text{ for all } x \in \mathbb{R}\} \). The function \( f(x) = -x^2 \in V \) but \( -(f) \not\in V \) so \( V \) is not a vector space.
   c) The set of all \( f \) of the form \( k_1 + k_2 \sin(x) \) where \( k_1, k_2 \) are real numbers is easily seen to be \( \text{span}\{1, \sin(x)\} \) and so is a vector space.

4. Let \( n \) be given. Which of the following are subspaces of the vector space of all \( n \times n \) matrices whose entries are real numbers.
   a) Let \( V = \{A = (a_{ij}) : \text{tr}(A) = 0\} \). We have \( A = (a_{ij}) \in V \) if \( a_{11} + a_{22} + \cdots + a_{nn} = 0 \). Similarly we have \( B = (b_{ij}) \in V \) if \( b_{11} + b_{22} + \cdots + b_{nn} = 0 \). Now \( \text{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn}) = a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0 \) and also \( \text{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0 \). Thus \( V \) is a vector space.
   b) Let \( V_B = \{A : AB = BA\} \). Then for \( A, C \in V \), we have \( (A + C)B = AB + CB = BA + BC = B(A + C) \) and \( (kA)B = k(AB) = k(BA) = B(kA) \). Hence \( V \) is a vector space.
c) The set of all \( n \times n \) matrices \( A \) such that the system of equations \( Ax = 0 \) has only the trivial solution \( x = 0 \) is the set of all \( n \times n \) matrices with non zero determinant. Now \( \det(I) = 1 \neq 0 \) and \( \det(-I) = (-1)^n \neq 0 \) yet \( I + (-I) = 0 \) and \( \det(0) = 0 \). Thus we do not have closure; the set of matrices is not a vector space.

5. Consider the two dimensional vector space \( V = \text{span}(\cos^2(x), \sin^2(x)) \), a subspace of all functions from \( \mathbb{R} \to \mathbb{R} \). Which of the following belong to \( V \) (the argument to show \( f \notin V \) will be more difficult).

(a) The zero function \( 0 \) is the \( 0 \) vector in the vector space of functions and is easily obtained as \( 0 \) times any function (in actual fact it is viewed as the outcome of the empty linear combination; a linear combination of no vectors) and so immediately \( 0 \in \text{span}(\cos^2(x), \sin^2(x)) \).

(b) We know \( 2 = 2(\cos^2(x) + \sin^2(x)) \in \text{span}(\cos^2(x), \sin^2(x)) \).

(c) We show that \( 3 + x^2 \notin \text{span}(\cos^2(x), \sin^2(x)) \) by showing that the three functions are linearly independent. If we evaluate the functions at \( 0, \pi/4, \pi/2 \), we check that

\[
\begin{vmatrix}
1 & 0 & 3 \\
1/2 & 1/2 & 3 + \pi^2/16 \\
0 & 1 & \pi^2/4
\end{vmatrix} = \pi^2/16 \neq 0
\]

This could be discovered by considering the very different growths of the functions, namely \( 3 + x^2 \) goes to infinity while the other functions have range in \([0, 1]\).

(d) We know \( \cos(2x) = \cos^2(x) - \sin^2(x) \in \text{span}(\cos^2(x), \sin^2(x)) \).

6. True or False (Give reasons!) If \( v_1, v_2, v_3 \) are non zero vectors and \( \{v_1, v_2, v_3\} \) are linearly dependent then each vector in the set is expressible as a linear combination of the other two.

False: Take \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \). The last vector is not a linear combination of the other two. For the conclusion to be true it would suffice to have a linear dependency that has non zero coefficients for each of the three vectors.

Other examples may arise with \( v_1 = 0 \).

7. Assume that 1 and \( \sqrt{2} \) are linearly dependent and there does exist 4 integers \( a, b, c, d \) with \( b \neq 0, d \neq 0 \) and not both \( a = 0 \) and \( c = 0 \), where in addition \( \gcd(a, b) = \gcd(c, d) = 1 \). which satisfy

\[
a/b \times 1 + c/d \times \sqrt{2} = 0.
\]

Assume that \( c \neq 0 \) and then \( x\sqrt{2} = -\frac{ad}{bc} = \frac{p}{q} \) for some integers \( p, q \) with \( \gcd(p, q) = 1 \). Squaring both sides, yields \( 2 = \frac{p^2}{q^2} \). If you know the unique factorization of integers as a product of primes, we immediately have that the power of 2 in \( p^2 \) is even as is the power of 2 in \( q^2 \) and so we have a contradiction. Alternately, we write \( 2q^2 = p^2 \) and so \( p^2 \) is even and so \( p \) is even, namely \( p = 2r \) for some integer \( r \). But then \( 2q^2 = (2r)^2 = 4r^2 \), and so \( q^2 \) must be even and so \( q \) is even. This violates are assumption that \( \gcd(p, q) = 1 \). Thus \( p, q \) can’t exist and so \( a, b, c, d \) can’t exist.

Interestingly you can track down proofs of the irrationality of \( \sqrt{2} \) by googling ‘square root of 2’ and getting a wikipedia article.

8. Continuing question 5 from assignment 2, compute \( e^{xS} \). We have

\[
S^k = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}^k = \begin{bmatrix}
\lambda^k & k\lambda^{k-1} \\
0 & \lambda^k
\end{bmatrix}
\]

We can now compute

\[
e^{tS} = I + tS + \frac{1}{2!}t^2S^2 + \frac{1}{3!}t^3S^3 + \cdots
\]
\[ I + t \left[ \begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array} \right] + \frac{1}{2!} t^2 \left[ \begin{array}{cc} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{array} \right] + \frac{1}{3!} t^3 \left[ \begin{array}{cc} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{array} \right] + \cdots \]
\[ = \left[ 1 + t\lambda + \frac{1}{2!} t^2 \lambda^2 + \cdots \right] \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] + t^2 \lambda + \frac{1}{2!} t^3 \lambda^2 + \cdots \]
\[ = \left[ e^{t\lambda} \begin{array}{c} 1 \\ 0 \end{array} \right] \]

9. Let \( J \) denote the \( n \times n \) matrix of all 1’s. We wish to compute \( \det(aJ + bI) \).

If we apply gaussian elimination to \( Jv = 0 \) (seeking eigenvectors of eigenvalue 0) we quickly find there is one corner variable and \( n - 1 \) free variables and so it is natural to choose one eigenvector associated with each free variable.

\[ v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad v_{n-1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

We have chosen these vectors so that by Gaussian Elimination techniques we have \( \{x : Jx = 0\} = \text{span}\{v_1, v_2, \ldots, v_{n-1}\} \).

Let \( v_n = 1 \) denote the vector of \( n \) 1’s. Note that \( Jv_n = nv_n \) so that \( v_n \) is an eigenvector of \( J \) of eigenvalue \( n \).

Assume \( \{v_1, v_2, \ldots, v_n\} \) are linearly dependent namely there are coefficients \( a_1, a_2, \ldots, a_n \) not all zero so that

\[ \sum_{i=1}^{n} a_i v_i = 0 = \sum_{i=1}^{n-1} a_i v_i + a_n v_n \]

Now \( \sum_{i=1}^{n-1} a_i v_i \) is an eigenvector \( u \) of eigenvalue 0 (if the sum \( \neq 0 \)) and \( a_n v_n \) is an eigenvector \( v \) of eigenvalue \( n \) (if \( a_n v_n \neq 0 \)). By our previous observations (assignment 2, question 1b) we cannot have \( u + v = 0 \). Thus both \( u \) and \( v \) are 0. If \( u = 0 \), then by the linear independence of \( v_1, v_2, \ldots, v_{n-1} \) we obtain \( a_1 = a_2 = \cdots = a_{n-1} \). If \( v = 0 \), then since \( v_n \neq 0 \), then \( a_n = 0 \). This is a contradiction. We now deduce that \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent. and hence if we form the matrix \( M = [v_1 v_2 \cdots v_n] \) then \( M \) is invertible. If we let \( D \) be the diagonal matrix with \( n-1 \) 0’s on the diagonal and one 1 in the final row and column then \( AM = MD \).

We now apply the ideas from our midterm question or otherwise to compute \( \det(aJ + bI) \) as follows \( \det(aJ + bI) = \det(aM^{-1}DM + bI) = \det(M^{-1}(aD + bI)M) = \det(aD + bI) \). Thus \( aD + bI \) has entries \( a \cdot 0 + b \cdot 1 \) on \( n-1 \) of the diagonal entries and \( a \cdot n + b \cdot 1 \) on the last diagonal entry.

Hence \( \det(aJ + bI) = \det(aD + bI) = b^{n-1}(an + b) \).

Note that the question 1 iii) is the \( 4 \times 4 \) case of \( \det(J + I) \).

10. We repeat assignment 3, question 8 for a larger matrix. You may restrict to the \( 3 \times 3 \) case. Let \( A \) be a \( 3 \times 3 \) matrix and let \( u \) be a vector. Let

\[ A^n u = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \]

Assume there is a vector \( v \) with

\[ \lim_{n \to \infty} \frac{A^n u}{x_n} = v. \]
(this requires \(x_n \neq 0\); there are ways to handle \(x_n = 0\) which we shall ignore here). Show that \(v\) is an eigenvector of \(A\). You may assume that \(u, v\) are linearly independent and that there is a third vector \(w\) with \(M = [u \ v \ w]\) being invertible and so \(u, v, w\) are linearly independent (i.e. we can extend \(u, v\) to a basis). You can consider the linear transformation \(f(x) = Ax\) with respect to this new basis \([u \ v \ w]\).

Following the proof we gave for this question in the \(2 \times 2\) case, we start with expressing \(A^n u\) as a linear combination of \(u, v, w\). Thus

\[
\begin{bmatrix}
x_n \\
y_n \\
z_n 
\end{bmatrix} = M \begin{bmatrix}
u_n \\
v_n \\
w_n 
\end{bmatrix}, \quad \begin{bmatrix}
u_n \\
v_n \\
w_n 
\end{bmatrix} = M^{-1} \begin{bmatrix}
x_n \\
y_n \\
z_n 
\end{bmatrix}
\]

We note that \(u_n, w_n\) must be small compared to \(v_n\) (for large \(n\)) in order that \(A^n u = u_n u + v_n v + w_n w\) satisfy the desired limit. In particular \(v_n \neq 0\) for large \(n\). Also \(|u_n/v_n|\) and \(|w_n/v_n|\) can be chosen to be suitably small for \(n\) large. Or perhaps it is easier to write \(\lim_{n \to \infty} u_n/v_n = \lim_{n \to \infty} w_n/v_n = 0\). Now compute \(A^{n+1} u = A(A^n u) = A(u_n u + v_n v + w_n w)\). We write \(A\) with respect to the vectors \([u, v, w]\). So we write \(A = MBM^{-1}\). We think that \(B\) is the same linear transformation as \(A\) but written in blue coordinates. See the online notes on white/blue coordinates. Now let

\[
B = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i 
\end{bmatrix}
\]

We have \(A(u_n u + v_n v + w_n w) = (au_n + bv_n + cw_n)u + (du_n + ev_n + fw_n)v + (gu_n + hv_n + iw_n)w\). Thus \(u_{n+1} = au_n + bv_n + cw_n, v_{n+1} = du_n + ev_n + fw_n\) and \(w_{n+1} = gu_n + hv_n + iw_n\). We would like to show \(b = h = 0\) which would show that \(v\) is an eigenvector of eigenvalue \(e\). We must have \(|u_{n+1}/v_{n+1}| = |(au_n + bv_n + cw_n)/(du_n + ev_n + fw_n)|\) small for large \(n\) but dividing top and bottom by \(v_n\) yields something close to the ratio \(b/e\). Now if \(b, e \neq 0\), then \(b/e\) is “far away” from 0 (essentially \(b/e\) different from 0) but \(|(au_n + bv_n + cw_n)/(du_n + ev_n + fw_n)|\) is small for large \(n\). Thus \(b = 0\). Similarly we must have \(|w_{n+1}/v_{n+1}| = |(gu_n + hv_n + iw_n)/(du_n + ev_n + fw_n)|\) small for large \(n\) and so \(h = 0\). But now \(Av = ev\) and so \(v\) is an eigenvector of eigenvalue \(e\).

It is easiest to think limits. We have

\[
\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{w_n}{v_n} = \lim_{n \to \infty} \frac{au_n + bv_n + cw_n}{du_n + ev_n + fw_n} = \lim_{n \to \infty} \frac{gu_n + hv_n + iw_n}{du_n + ev_n + fw_n} = 0
\]

Now we compute

\[
\lim_{n \to \infty} \frac{au_n + bv_n + cw_n}{du_n + ev_n + fw_n} = \lim_{n \to \infty} \frac{a u_n}{d u_n} + b + c \frac{w_n}{w_n} = \frac{b}{e} = 0
\]

Thus we conclude \(b = 0\). Similarly \(h = 0\) and so \(v\) is an eigenvector of eigenvalue \(e\).

11. This is a harder contest type problem. Let \(A\) be a \(2013 \times 2014\) matrix of integer entries such that each row sum is 0 (i.e. \(A \mathbf{1} = \mathbf{0}\) where \(\mathbf{1}\) is the \(2013 \times 1\) vector of 1’s and \(\mathbf{0}\) is the \(2013 \times 1\) vector of 0’s). Show that \(\det(AA^T) = 2014k^2\) for some integer \(k\).

Here is a solution. Form a new \(2014 \times 2014\) matrix \(B\) from \(A\) by adding a row of 2014 1’s to the bottom of \(A\). The row of 1’s dot any row of \(A\) is 0 and so we find that \(BB^T\) is

\[
\begin{bmatrix}
AA^T & \mathbf{0} \\
\mathbf{0}^T & 2014
\end{bmatrix}
\]
Thus $\det(BB^T) = 2014 \times \det(AA^T)$ Now $\det(BB^T) = (\det(B))^2$. Thus $(\det(B))^2 = 2014 \times \det(AA^T)$. Now $\det(B)$ is an integer. In any factorization of an integer squared, each prime will appear an even number of times. Noting that $2014 = 2 \times 19 \times 53$, we deduce that $\det(AA^T)$ must be $2014k^2$ for some integer $k$.

I had a much more involved proof that also works that I will post on our website. The hint makes this solution seem the best.