1. Given the three vectors
\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]
consider each of the 6 possible orderings of the vectors as three columns of a $3 \times 3$ matrix. Show that the determinant is $+1$ if the ordering of three vectors obeys the right hand rule and the determinant is $-1$ if the ordering of three vectors obeys the left hand rule (see https://en.wikipedia.org/wiki/Right-hand_rule concerning coordinate systems).

2. Which of the following are subspaces of $\mathbb{R}^3$?
   a) all vectors of the form $(a, 0, 0)$.
   b) all vectors of the form $(a, b, c)$ where $c = a + b$.

3. Which of the following are subspaces of the vector space of all functions $f$ with domain $\mathbb{R}$ and range contained in $\mathbb{R}$?
   a) all $f$ such that $f(-1) = 0$.
   b) all $f$ such that $f(x) \leq 0$ for all $x \in \mathbb{R}$.
   c) all $f$ of the form $f(x) = k_1 + k_2 \sin(x)$ where $k_1, k_2 \in \mathbb{R}$.

4. Let $n$ be given. Which of the following are subspaces of the vector space of all $n \times n$ matrices whose entries are real numbers.
   a) all $n \times n$ matrices such that $\text{tr}(A) = 0$.
   b) all $n \times n$ matrices $A$ such that for a given fixed matrix $B$, we have $AB = BA$.
   c) all $n \times n$ matrices $A$ such that the system of equations $Ax = 0$ has only the trivial solution $x = 0$.

5. Consider the two dimensional vector space $V = \text{span}(\cos^2(x), \sin^2(x))$, a subspace of all functions from $\mathbb{R} \to \mathbb{R}$. Which of the following belong to $V$ (the argument to show $f \notin V$ will be more difficult).
   (a) $0$ (b) $2$ (c) $3 + x^2$ (d) $\cos(2x)$

6. True or False (Give reasons!) If $v_1, v_2, v_3$ are non zero vectors and $\{v_1, v_2, v_3\}$ are linearly dependent then each vector in the set is expressible as a linear combination of the other two.

7. Show that $1$ and $\sqrt{2}$ are linearly independent when we restrict ourselves to the scalar field $\mathbb{Q}$, the rational numbers. In other words show that there do not exist 4 integers $a, b, c, d$ with $b \neq 0$, $d \neq 0$ and not both $a = 0$ and $c = 0$, which satisfy
   \[ \frac{a}{b} \times 1 + \frac{c}{d} \times \sqrt{2} = 0. \]

8. Compute $e^{tS}$ for
   \[ S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \].

9. Let $J$ denote the $n \times n$ matrix of all 1’s. We wish to compute $\det(aJ + bI)$, namely the determinant of the matrix with $a$’s off the diagonal and $(a + b)$’s on the diagonal. Determine $n - 1$ eigenvectors $v_1, v_2, \ldots, v_{n-1}$ of $J$ of eigenvalue 0 so that every other eigenvector of eigenvalue 0 is a linear combination of $v_1, v_2, \ldots, v_{n-1}$. Use our standard gaussian elimination to do this. Let $v_n = 1$ denote the vector of $n$ 1’s. Note that $Jv_n = nv_n$ so that $v_n$ is an eigenvector of $J$ of eigenvalue $n$. 

Form an \( n \times n \) matrix \( M = [v_1 v_2 \ldots v_n] \) whose columns are the eigenvectors. Argue that \( M \) is invertible. Perhaps Assignment 3, question 6 is helpful. Show that the eigenvalues of \( J \) are eigenvalue 0 with multiplicity \( n - 1 \) (as a root of \( \det(J - \lambda I) \)) and eigenvalue \( n \) with multiplicity 1. Assignment 4, question 5 may be helpful.

Having shown that \( J \) is diagonalizable, now compute \( \det(aJ + bI) \).

10. We repeat assignment 3, question 8 for a larger matrix. You may restrict to the \( 3 \times 3 \) case. Let \( A \) be a \( 3 \times 3 \) matrix and let \( u \) be a vector. Let

\[
A^n u = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}
\]

Assume there is a vector \( v \) with

\[
\lim_{n \to \infty} \frac{A^n u}{x_n} = v.
\]

(this requires \( x_n \neq 0 \); there are ways to handle \( x_n = 0 \) which we shall ignore here). Show that \( v \) is an eigenvector of \( A \). You may assume that \( u, v \) are linearly independent and that there is a third vector \( w \) with \( \begin{bmatrix} u & v & w \end{bmatrix} \) being invertible and so \( u, v, w \) are linearly independent (i.e. we can extend \( u, v \) to a basis). You can consider the linear transformation \( f(x) = Ax \) with respect to this new basis \( \{u v w\} \).

11. This is a harder contest type problem. Let \( A \) be a \( 2013 \times 2014 \) matrix of integer entries such that each row sum is 0 (i.e. \( A1 = 0 \) where \( 1 \) is the \( 2014 \times 1 \) vector of 1’s and \( 0 \) is the \( 2013 \times 1 \) vector of 0’s. Show that \( \det(AA^T) = 2014k^2 \) for some integer \( k \).

Hint: You might find it helpful to form a new square matrix \( B \) from \( A \) by adding a row of 1’s. What is \( \det(BB^T) \)?