1. Show that the $2 \times 2$ matrix of 0’s is diagonalizable (I mention this because many students confuse invertibility and diagonalizability; they are not related).

2. Let $A_k$ denote the $2 \times 2$ matrix

$$A_k = \begin{bmatrix} k & 1 \\ -1 & 3 \end{bmatrix}$$

For what values of $k$ is $A_k$ diagonalizable? Namely, can we find two eigenvectors $u, v$ of $A_k$ so that if we let $M = [u \ v]$ then $M$ is invertible? Use 2(b) from assignment 2 to handle the case of two different eigenvalues; you need not explicitly find the eigenvectors in that case.

I wish to see the solutions to a system of equations in Parametric Vector Form (or Vector Parametric Form). For example if the set of solutions is:

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = 1/3$$

for all choices $r, s, t \in \mathbb{R}$ then we can write the set of solutions in parametric vector form as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad r, s, t \in \mathbb{R}$$

3. Give the solutions in vector parametric form for the plane $\pi = \{(x, y, z) : 2x - 2y + 3z = 5\}$.

4. Give the vector parametric form of all solutions to the following system of equations:

$$2x_1 + 4x_4 + 6x_5 = 14$$
$$2x_1 + 5x_4 + 7x_5 = 16$$
$$3x_1 + 2x_2 + 8x_4 + 9x_5 = 27$$
$$3x_1 + 4x_2 + 13x_4 + 12x_5 = 39$$

5. Express the line given by the vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad s \in \mathbb{R}$$

as the set of solutions to a system of equations in $x, y, z$ (two equations suffice; eliminate $s$). Then use Gaussian Elimination on this system of equations in $x, y, z$ to re-express the solutions in vector parametric form.

6. Express the solutions to

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$
in vector parametric form as \( \mathbf{x} = \mathbf{a} + s \mathbf{b} + t \mathbf{c} + u \mathbf{d} \). Show that the only solution to

\[
\begin{bmatrix}
    b & c & d
\end{bmatrix}
\begin{bmatrix}
    s \\
    t \\
    u
\end{bmatrix} = 0 \quad \text{is} \quad \begin{bmatrix}
    s \\
    t
\end{bmatrix} = 0.
\]

7. Express the inverse of the following matrix \( A \) as a product of elementary matrices and use this product to express the matrix \( A \) as a product of elementary matrices.

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
2 & 3 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

8. Let \( A \) be a \( 2 \times 2 \) matrix with first column \( \mathbf{x} \) and second column \( \mathbf{y} \). We wish to show that \( |\det(A)| \) is the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \). Using assignment 2, question 2 (a), we readily deduce that \( |\det(A)| = 0 \) if and only if the parallelogram is degenerate with no area. So assume \( |\det(A)| \neq 0 \).

a) There exists an angle \( \theta \) such that \( R(\theta) \mathbf{x} = \mathbf{x}' \) points in the direction of the \( x \)-axis. Let \( \mathbf{y}' = R(\theta) \mathbf{y} \). Explain why the area of the parallelogram formed by \( \mathbf{x}', \mathbf{y}' \) is the same as the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \).

b) We can choose a value \( s \) so that if we set \( S = \begin{bmatrix}
1 & s \\
0 & 1
\end{bmatrix} \), then this shear matrix has \( S \mathbf{x}' = \mathbf{x}' \) while \( S \mathbf{y}' = \mathbf{y}'' \) has \( \mathbf{y}'' \) in the direction of the \( y \)-axis or its opposite. Explain why the area of the parallelogram formed by \( \mathbf{x}', \mathbf{y}'' \) is the same as the area of the parallelogram formed by the two vectors \( \mathbf{x}', \mathbf{y}' \).

c) If we let \( B \) be a \( 2 \times 2 \) matrix with first column \( \mathbf{x}' \) and second column \( \mathbf{y}'' \), Explain why \( |\det(B)| \) is the area of the rectangular box formed by \( \mathbf{x}', \mathbf{y}'' \).

d) Using the product rule for determinants (\( \det(EF) = \det(E) \det(F) \) for any pair of \( 2 \times 2 \) matrices) and verifying that \( \det(R(\theta)) = 1 \) and \( \det(S) = 1 \), show that \( |\det(A)| \) is the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \).

9. This is harder than question 7 from assignment 2 and essentially is the reverse observation. Let \( A \) be a \( 2 \times 2 \) matrix (not necessarily diagonalizable) and define

\[
\begin{bmatrix}
    x_n \\
    y_n
\end{bmatrix} = A^n \begin{bmatrix}
    1 \\
    2
\end{bmatrix}.
\]

Assume \( \lim_{n \to \infty} \frac{x_n}{y_n} = 1 \). Show that \( \begin{bmatrix}
    1 \\
    1
\end{bmatrix} \) is an eigenvector of \( A \).

**Hint:** This is one possible approach. Explain why we can find \( c_1, c_2, c_3, c_4 \) with

\[
A \begin{bmatrix}
    1 \\
    1
\end{bmatrix} = c_1 \begin{bmatrix}
    1 \\
    1
\end{bmatrix} + c_2 \begin{bmatrix}
    1 \\
    2
\end{bmatrix}; \quad A \begin{bmatrix}
    1 \\
    2
\end{bmatrix} = c_3 \begin{bmatrix}
    1 \\
    1
\end{bmatrix} + c_4 \begin{bmatrix}
    1 \\
    2
\end{bmatrix}.
\]

In effect this rewrites the matrix \( A \) with respect to the two vectors \( \begin{bmatrix}
    1 \\
    1
\end{bmatrix}, \begin{bmatrix}
    1 \\
    2
\end{bmatrix} \). What do you need to show about \( c_1, c_2, c_3, c_4 \) so that \( \begin{bmatrix}
    1 \\
    1
\end{bmatrix} \) is an eigenvector of \( A \)?

**Show that we can write**

\[
A^n \begin{bmatrix}
    1 \\
    2
\end{bmatrix} = \frac{x_n}{y_n} \quad \text{as} \quad a \begin{bmatrix}
    1 \\
    1
\end{bmatrix} + b \begin{bmatrix}
    1 \\
    2
\end{bmatrix}.
\]

**Can you say anything about** \( a, b \) ?

**Now compare** \( A^n \begin{bmatrix}
    1 \\
    2
\end{bmatrix} \) **with** \( A^{n+1} \begin{bmatrix}
    1 \\
    2
\end{bmatrix} = A \left( A^n \begin{bmatrix}
    1 \\
    2
\end{bmatrix} \right) \).