1. We are asked to apply Gram-Schmidt to the three vectors:

\[
\begin{bmatrix}
1 \\
2 \\
-2
\end{bmatrix}, \quad
\begin{bmatrix}
4 \\
3 \\
2
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}.
\]

We set \( u_1 = (1, 2, -2)^T \) and then set

\[
u_2 = (4, 3, 2)^T - \text{proj}_{(1,2,-2)^T}(4, 3, 2)^T = (10/3, 5/3, 10/3)^T.
\]

For convenience, let \( u_2 = (2, 1, 2)^T \). Now

\[
u_3 = (1, 2, 1)^T - \text{proj}_{(1,2,-2)^T}(1, 2, 1)^T - \text{proj}_{(2,1,2)^T}(1, 2, 1) = (-2/3, 2/3, 1/3)^T.
\]

Normalizing, we get an orthonormal basis

\[
\begin{bmatrix}
1/3 \\
2/3 \\
-2/3
\end{bmatrix}, \quad
\begin{bmatrix}
2/3 \\
1/3 \\
2/3
\end{bmatrix}, \quad
\begin{bmatrix}
-2/3 \\
2/3 \\
1/3
\end{bmatrix}.
\]

Question 2 is likely (very likely) to be similar to a question on the final. The others are all related to exam questions. Note that we will prove that an \( n \times n \) symmetric matrix always is diagonalizable with an orthonormal basis of eigenvectors for \( \mathbb{R}^n \).

2. Find orthonormal bases of eigenvectors for the following matrices (you can find such questions on every exam):

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad
B = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}, \quad
C = \begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}
\]

Hint: for \( B \), 0 is an eigenvalue, and for \( C \), -2 is an eigenvalue.

A: \( \det (A - \lambda I) = \lambda^2 - 5\lambda = \lambda(\lambda - 5). \) For eigenvalue \( \lambda = 0 \), we obtain an eigenvector \((2,1)^T\). For eigenvalue \( \lambda = 5 \), we obtain an eigenvector \((1,2)^T\). No need to apply Gram Schmidt since the vectors are already orthogonal. We obtain an orthonormal basis of eigenvectors by merely normalizing them: \((-2/\sqrt{5}, 1/\sqrt{5})^T \) and \((1/\sqrt{5}, 2/\sqrt{5})^T \).

B: \( \det (B - \lambda I) = -((\lambda)(\lambda - 15))(\lambda + 15). \) For eigenvalue \( \lambda_1 = 0 \), we obtain an eigenvector \((1/2, -1, 1)^T\). For eigenvalue \( \lambda_2 = 15 \), we obtain an eigenvector \((2, 2, 1)^T\). For eigenvalue \( \lambda_3 = -15 \), we obtain an eigenvector \((-1, 1/2, 1)^T\). We obtain an orthonormal basis of eigenvectors by merely normalizing them:

\[
(1/3, -2/3, 2/3)^T, \quad (2/3, 2/3, 1/3)^T, \quad (-2/3, 1/3, 2/3)^T,
\]

C: \( \det (C - \lambda I) = -((\lambda + 2)^2(\lambda - 4). \) For eigenvalue \( \lambda_1 = 4 \), we obtain an eigenvector \((1, -2, 1)^T\). For eigenvalue \( \lambda_2 = -2 \), we obtain a two dimensional eigenspace with a basis of eigenvectors \((-1, 0, 1)^T \) and \((2, 1, 0)^T \). Apply Gram-Schmidt to the second vector to replace it by \((2, 1, 0)^T - \text{proj}_{(-1,0,1)^T}(2, 1, 0)^T = (1, 1, 1)^T \). We can now report an orthonormal basis of eigenvectors by normalizing them:

\[
\frac{1}{\sqrt{6}}(1, -2, 1)^T, \quad \frac{1}{\sqrt{2}}(-1, 0, 1)^T, \quad \frac{1}{\sqrt{3}}(1, 1, 1)^T.
\]
3. (adapted from an exam) Let $A$ be a $3 \times 3$ matrix with $\det(A - \lambda I) = -(\lambda^3 + a\lambda^2 + b\lambda + c)$. Show that if $A$ is diagonalizable, then the following equation is true

$$A^3 + aA^2 + bA + cI = 0$$

(this equation is in fact true for any $3 \times 3$ matrix and is a special case of the Cayley-Hamilton Theorem).

4. Let $\{u_1, u_2, \ldots, u_k\}$ be non-zero vectors satisfying $u_i \cdot u_j = 0$ for all pairs $i, j$ with $i \neq j$. Show that $\{u_1, u_2, \ldots, u_k\}$ are linearly independent.

Assume $a_1u_1 + a_2u_2 + \cdots + a_ku_k = 0$. We multiply both sides on the left by $u_i^T$ (for some $1 \leq i \leq k$) to obtain

$$u_i^T(a_1u_1 + a_2u_2 + \cdots + a_ku_k) = \sum_{j=1}^k a_ju_i^T u_j = a_i(u_i^Tu_i) = u_i^T0 = 0.$$  

Using $u_i \neq 0$ so that $u_i^Tu_i > 0$, we deduce $a_i = 0$. But this is true for each choice of $i$ with $1 \leq i \leq k$ and so $\{u_1, u_2, \ldots, u_k\}$ are linearly independent.

5. (from an exam) Let $A$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may repeat) and an orthonormal basis of eigenvectors $u_1, u_2, \ldots, u_n$ ($Au_i = \lambda_iu_i$). Then show that

$$A = \sum_{i=1}^n \lambda_iu_iu_i^T.$$  

(thus $A$ is a sum of $n$ symmetric rank 1 matrices)

Let $M$ be the matrix with $ith$ column equal to $u_i$ so that $M^T = M^{-1}$.

Consider the matrix equation $A = (MD)M^T$. The matrix $MD$ has $ith$ column equal to $\lambda_iu_i$. Now as noted in the website file on matrix multiplication, if we have a matrix product $BC$ of two $n \times n$ matrices, then it is the sum of $n$ rank 1 matrices formed by the $ith$ column of $B$ times the $1 \times n$ matrix of the $ith$ row of $C$. This implies $A = (MD)M^T = \sum_{i=1}^n \lambda_iu_iu_i^T$.

An alternate approach is to verify that if we set $B = \sum_{i=1}^n \lambda_iu_iu_i^T$, then $Au_i = Bu_i$ for $i = 1, 2, \ldots, n$ and hence $AM = BM$ and then multiplying on the right by $M^{-1}$ we obtain $A = B$. Thus we compute $Au_i = \lambda_iu_i$. We check that $u_ju_j^Tu_i = u_j(u_j \cdot u_i)$ which is $0$ if $i \neq j$ (because $u_j$ and $u_i$ are orthogonal) and which is $u_i$ if $i = j$ (because $u_i$ is of unit length). Hence $Bu_i = \sum_{j=1}^n \lambda_ju_ju_j^Tu_i = \lambda_iu_i$. Thus as argued above, $A = B$.

Another approach started by writing the diagonal matrix $D$ as $D = \sum_{i=1}^n \lambda_i e_i e_i^T$ and then applying $A = DMD^T$ to have $A = M(\sum_{i=1}^n \lambda_i e_i e_i^T)M^T = \sum_{i=1}^n \lambda_i(Me_i)(Me_i)^T$ to obtain our result.

6. (from an exam) Consider a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We could attempt to solve for $A^{-1}$ by letting

$$A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

with four variables $x, y, z, t$ and then $AA^{-1} = I$ becomes a system of equations. What is the rank of the $4 \times 4$ system of equations assuming $A^{-1}$ exists? Explain. Can you say anything about the rank if $\det(A) = 0$? Explain.

The system of equations (using $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z & t \end{bmatrix} = I$) is the matrix equation

$$\begin{bmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
Thus if the rank of the $4 \times 4$ system of equations is 4, we can solve uniquely for $A^{-1}$ and if the rank is less than 4 then there will be free variables and so if the system were consistent then the inverse would not exist, which is impossible and so if $A^{-1}$ exists, the rank is 4. Moreover, if $\det(A) = 0$ and so $A^{-1}$ does not exist then the rank is less than 4. A more careful analysis, shows that the rank of the $4 \times 4$ system is either 0, 2 or 4, and is in fact twice the rank of $A$.

7. (from an exam) You are attempting to solve for $x, y, z$ in the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Find a ‘least squares’ choice $\hat{b}$ in the column space of $A$ (and hence with $||b - \hat{b}||^2$ being minimized) and then solve the new system $Ax = \hat{b}$ for $x, y, z$. You can check your choice of $\hat{b}$ by testing if $b - \hat{b}$ is orthogonal to the column space of $A$.

The columns of $A$ are already orthogonal so we can easily form an orthonormal basis for $\text{colsp}(A)$.

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus we compute $\hat{b}$ by projecting orthogonally into $\text{colsp}(A)$. Thus

$$\hat{b} = (b \cdot u_1)u_1 + (b \cdot u_2)u_2 + (b \cdot u_3)u_3 = 3u_1 - 2u_2$$

Thus

$$\hat{b} = 3 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - 2 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solving the system $Ax = \hat{b}$ we can read off carefully that since $3u_1 - 2u_2 = \hat{b}$, and $u_1, u_2$ are both half the respective columns in $A$, we deduce that $x = (3/2, -1, 0)^T$.

We check that $b - \hat{b} = (1/2, -1/2, 1/2, -1/2)^T$ and this is orthogonal to each of $u_1, u_2, u_3$ and so is orthogonal to all vectors in $\text{colsp}(A)$. I made up this question from the special matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

which is known as a Hadamard matrix.

8. (harder) You are given the following vector recurrence for $n \geq 1$.

$$x_n = b + Ax_{n-1}$$

where $x_0$ has been given to you to begin the recurrence. We are given that $A$ is a $3 \times 3$ matrix with eigenvalues $|\lambda_3| < |\lambda_2| < \lambda_1 = 1$,
and associated eigenvectors $v_3, v_2, v_1$. Assume $x_0 = x_1v_1 + x_2v_2 + x_3v_3$ and $b = b_1v_1 + b_2v_2 + b_3v_3$ with $x_1 \neq 0$ and $b_1 \neq 0$. Show that
\[
\lim_{n \to \infty} x_n - x_{n-1} = b_1 v_1.
\]

We compute \( x_1 = b + Ax_0; \) \( x_2 = b + (A(b + Ax_1)) = b + A(b + Ax_0) = b + Ab + A^2x_0, \)
\( x_3 = b + Ab + A^2b + A^3x_0 \) and so
\[
x_t = b + Ab + A^2b + \cdots + A^{t-1}b + A^t x_0
\]

We then compute
\[
x_n - x_{n-1} = b + Ab + A^2b + \cdots + A^{n-1}b + A^n x_0 - (b + Ab + A^2b + \cdots + A^{n-2}b + A^{n-1}x_0)
\]
\[= A^{n-1}b + A^n x_0 - A^{n-1}x_0\]

Now we readily deduce that \( A^t x_0 = A^t(x_1v_1 + x_2v_2 + x_3v_3) = x_1v_1 + (x_2\lambda_2^t v_2 + x_3\lambda_3^t v_3). \) Since \( |\lambda_3| < |\lambda_2| < 1 \) we have \( \lim_{n \to \infty} (\lambda_2)^t = 0 = \lim_{n \to \infty} (\lambda_3)^t. \) Thus \( \lim_{n \to \infty} A^t x_0 = x_1 v_1. \) Similarly \( \lim_{n \to \infty} A^t b = b_1 v_1. \)

This problem is a case of Markov chains. The one dimensional space of eigenvectors of eigenvalue 1 with all other eigenvalues of smaller magnitude in absolute value is an important property.