Math 223  Assignment 8  Due Friday Nov 20 in class.

1. Solve the differential equation given the initial conditions $x_1(0) = -3$, $x_2(0) = 4$.

\[ \frac{d}{dt}x_1(t) = x_2(t) \]
\[ \frac{d}{dt}x_2(t) = -2x_1(t) - 2x_2(t) \]

You compute

\[ \det \left[ \begin{array}{cc} 0 - \lambda & -1 \\ -2 - \lambda & -2 \\ \end{array} \right] = \lambda^2 + 2\lambda + 2 \]

and find the roots as $-1 + i$ and $-1 - i$ (complex eigenvalues come in conjugate pairs because the matrix has real entries).

\[ \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ -\frac{1}{2} + \frac{1}{2}i \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ -\frac{1}{2} + \frac{1}{2}i \end{bmatrix} \begin{bmatrix} -1 + i & 0 \\ 0 & -1 - i \end{bmatrix} \]

We use the initial conditions to solve for $c_1, c_2$ in

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{(-1+i)t} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} \]

So we have

\[ \begin{bmatrix} -3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} \]

from which we obtain $c_1, c_2$ using

\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{1}{2}i \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \]

Thus $c_1(t) = 2 - i$ and $c_2 = 2 + i$. We then substitute to obtain

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (2 - i)e^{(-1+i)t} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + (2 + i)e^{(-1-i)t} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} \]

Thus for example the first term, top entry is

\[ (2 - i)e^{-t}((\cos(t) + i\sin(t))(-\frac{1}{2} - \frac{1}{2}i)) \]
\[ = e^{-t}\left(-\frac{3}{2}\cos(t) + i\frac{1}{2}\sin(t)\right) \]

The other term is the complex conjugate and so we obtain the solution $x_1(t) = e^{-t}(-3\cos(t) + \sin(t))$. Similarly the first term, bottom entry is

\[ (2 - i)e^{-t}((\cos(t) + i\sin(t))(-\frac{1}{2} - \frac{1}{2}i)) \]
\[ = e^{-t}(2\cos(t) + i\sin(t)) \]

Thus the other term is the complex conjugate and so we obtain the solution $x_2(t) = e^{-t}(4\cos(t) + 2\sin(t))$. You can verify that these satisfy the differential equation (no complex numbers required!)
It would be easier to do this by computing the Real and Imaginary part of our solution (what $c_1$ multiplies) and then proceeding to solve for $c_1, c_2$.

2. You are given a 3 dimensional vector space $V \subseteq \mathbb{R}^5$. Could there be a $3 \times 6$ matrix $A$ with nullspace of $A$ being $V$? Explain. Could there be $6 \times 5$ matrix $B$ with nullspace of $B$ being $V$? Explain. In either case, given a basis for the three dimensional space $V$, how would you find the desired matrix assuming it exists.

In the first case null space$(A) \subseteq \mathbb{R}^6$ while $V \subseteq \mathbb{R}^5$ so this can never work. In the second case we want dim(null space$(B)) = 3 = 5$− rank$(B)$ and so rank$(B) = 2$. We have that dim$(V^\perp) = 5 – 3 = 2$ and so choose a basis for $V^\perp$, say $x, y$. Then form $B$ with 4 rows equal to $x^T$ and one row equal to $y^T$.

3. Consider the two planes $\pi_1$: $x - y + 2z = 3$ and $\pi_2$: $x + 2y + 3z = 6$. vskip 3pt a) Find the intersection of $\pi_1$ and $\pi_2$ in vector parametric form.

\[
\begin{bmatrix}
1 & -1 & 2 \\
1 & 2 & 3 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 2 \\
0 & 3 & 1 \\
\end{bmatrix}
\]

from which we read off the solutions as $(4, 1, 0)^T + s(-7/3, -1/3, 1)^T$ for $s \in \mathbb{R}$ (namely a line).

b) What is the angle (and cosine of the angle) formed by the two planes? This is most readily computed at the angle between the normals and hence $\cos \theta = (1, -2, 2)^T \cdot (1, 2, 3) / \| (1, -2, 2)^T \| \| (1, 2, 3)^T \| = 5 / (2 \sqrt{21})$. We compute $\theta = \cos^{-1}(5 / (2 \sqrt{21})) \approx 57^\circ$.

c) Find the distance of the point $(-1, 2, 2)$ to the plane $\pi_1$. We first note that $(3, 0, 0)^T \in \pi_1$ and then the length of the projection of $(-4, 2, 2)^T = (-1, 2, 2)^T - (3, 0, 0)^T$ onto the normal $(1, -1, 2)^T$, which is $-2/\sqrt{6}$ and so the distance is $\frac{\sqrt{2}}{6} \sqrt{6}$.

d) Find the equation of the plane parallel to $\pi_1$ through the point $(3, 2, 0)$. We simply take the plane to be $x - y + 2z = c$ for some constant $c$ and since $(3, 2, 0)$ is on the plane we take $c = 1$ so that the equation of the plane is $x - y + 2z = 1$.

e) Imagine the direction $(0, 0, 1)^T$ as pointing straight up from your current position $(0, 0, 0)^T$ in 3-space and the plane $\pi_2$ as a physical plane. If a marble is placed on $\pi_2$ at the point $(6, 0, 0)^T$, what direction will the marble roll under the influence of gravity? There are a variety of ways to approach this problem. Using forces, we know that the marble is acted upon by two forces: gravity in the direction $(0, 0, -1)^T$ and force upward by the plane in the direction of the normal $(1, 2, 3)^T$.

The resultant force is in the intersection of the plane formed by these two vectors and $\pi_1$ where we take the $z$ coordinate negative since we will be moving downhill. Alternatively, we can project the gravity vector $(0, 0, -1)^T$ into the plane by subtracting the projection of the vector $(0, 0, -1)^T$ onto the normal:

\[
\begin{bmatrix}
0 \\
-1
\end{bmatrix} - \frac{-3}{14} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
3/14 \\
6/14 \\
-5/14
\end{bmatrix}
\]

and so the marble rolls in the direction $(3, 6, -5)^T$.

4. Let $A$ and $B$ be similar matrices $n \times n$, namely there is an invertible matrix $M$ with $A = MBM^{-1}$. Show that $\text{tr}(A) = \text{tr}(B)$. Hint: Assignment 7, question 3.

We have that any degree $n$ polynomial factors into linear factors using the fundamental theorem of algebra. We have that

\[
\det(A - \lambda I) = \det(B - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda)
\]
We are allowing repeated roots. Now from assignment 7, we have that \( \text{tr}(A) = \sum_i \lambda_i \) and \( \text{tr}(B) = \sum_i \lambda_i \) hence \( \text{tr}(A) = \text{tr}(B) \).

An alternate solution is to show \( \text{tr}(AB) = \text{tr}(BA) \) for matrices \( A, B \) and then employ this to have \( \text{tr}(A) = \text{tr}(MBM^{-1}) = \text{tr}(M(BM^{-1})) = \text{tr}((BM^{-1})M) = \text{tr}(B) \). Showing \( \text{tr}(AB) = \text{tr}(BA) \) is just lots of summations

\[
\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji}a_{ij} = \text{tr}(BA)
\]

5. Let \( A \) be a \( n \times n \) matrix of real entries satisfying \( A^2 = -I \). Show that

a) \( A \) is nonsingular because \( AA = A^2 = -I \) and so \( (-A)(A) = -A^2 = -(-I) = I \) and so \( A^{-1} = -A \).

b) \( A \) has no real eigenvalues. If \( Ax = \lambda x \) for some \( x \neq 0 \), then \( A^2x = A(Ax) = A(\lambda x) = \lambda (Ax) = \lambda^2 x \). At the same time \( A^2x = (-I)x = -x \). Using \( x \neq 0 \). We deduce \( \lambda^2 = -1 \) and so there are no real values for \( \lambda \). In fact if \( \lambda \) is an eigenvalue then \( \lambda^2 = 1 \) and so \( \lambda = i \) or \( -i \).

c) \( n \) is even. We compute \( \det(A^2) = (\det(A))^2 = \det(-I) = (-1)^n \). We deduce that \( n \) is even. Otherwise if \( n \) is odd we have \( (\det(A))^2 = -1 \) at the same time \( \det(A) \) is real, which is a contradiction.

d) We immediately have \( \det(A) = \pm 1 \), from \( (\det(A))^2 = 1 \). But we need to show that \( \det(A) = 1 \).

There are two extra facts we can use. The product of the eigenvalues, using multiplicities of the roots, is \( \det(A) \), and the sum of the eigenvalues is \( \text{tr}(A) \). The eigenvalues are \( i \) or \( -i \), by our argument in (b). Let the multiplicity of \( i \) as a root of \( \det(A - \lambda I) \) be \( s \) and let the multiplicity of \( -i \) be \( t \), so that \( s + t = n \) (the polynomial \( \det(A - \lambda I) \) will factor into linear factors over \( \mathbb{C} \)). The sum of the eigenvalues is \( (s - t)i \). Thus \( \text{tr}(A) = (s - t)i \). But the trace of \( A \) is real, since the entries of \( A \) are real, so \( s = t \). But then \( \det(A) = (i)^s(-i)^t = (1)^s = 1 \) using the determinant as the product of the eigenvalues counted with multiplicity.

6. Consider two vector spaces \( U, V \), subspaces of \( \mathbb{R}^n \). Define \( U + V = \{ u + v : u \in U, v \in V \} \). Show that \( U + V \) is a vector space. Now show that

\[
\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V).
\]

(Hint: if we have an \( m \times n \) matrix \( A \) then \( n = \dim(\text{nulsp}(A)) + \text{rank}(A) \). How should we form \( A? \) )

We verify that \( U + V \) is a vector space by verifying closure. Let \( w_1, w_2 \in U + V \) with \( w_1 = u_1 + v_1 \) and \( w_2 = u_2 + v_2 \). Then \( w_1 + w_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2) \). Since \( U, V \) are vector spaces, then \( u_1 + u_2 \in U \) and \( v_1 + v_2 \in V \). Then \( w \in U + V \). Similarly \( kw_1 = k(u_1 + v_1) = ku_1 + kv_1 \).

Consider \( U \cap V \). If \( U \cap V = \{0\} \) then Let \( \mathcal{W} = \{ w_1, w_2, \ldots, w_s \} \) be a basis for \( U \cap V \). We can extend \( \{w_1, w_2, \ldots, w_s\} \) to a basis \( \mathcal{U} = \{u_1, u_2, \ldots, u_k\} \) for \( U \) and to a basis \( \mathcal{V} = \{v_1, v_2, \ldots, v_{\ell}\} \) for \( V \) where \( \mathcal{W} \subseteq \mathcal{U} \) and \( \mathcal{W} \subseteq \mathcal{V} \). Now we claim \( \mathcal{U} \cup \mathcal{V} = (\mathcal{U} \setminus \mathcal{W}) \cup \mathcal{W} \cup (\mathcal{V} \setminus \mathcal{W}) \) is a basis for \( U + V \). Given that the new set contains a basis for \( U \) and a basis for \( V \) we deduce that \( U + V \subseteq \text{span}(\mathcal{U} \cup \mathcal{V}) \). Now is \( \mathcal{U} \cup \mathcal{V} \) an independent set? For convenience assume the first \( s \) vectors of \( \mathcal{U} \) and \( \mathcal{V} \) are \( \mathcal{W} \).

Assume

\[
\sum_{i=s+1}^{k} a_i u_j + \sum_{i=s+1}^{\ell} b_i v_j + \sum_{i=1}^{s} c_i w_i = 0.
\]

Following from the linear independence of \( \mathcal{U} \) we deduce that if \( b_{s+1} = b_{s+2} = \cdots = b_{\ell} = 0 \), then \( c_1 = c_2 = \cdots = c_s = a_{s+1} = a_{s+2} = \cdots = a_{k} = 0 \). So we may assume \( b_j \neq 0 \) for some \( j \in \{s + 1, s + 2, \ldots, \ell\} \). Similarly we may deduce that \( a_p \neq 0 \) for some \( p \in \{s + 1, s + 2, \ldots, k\} \).
But now \( \sum_{i=s+1}^{\ell} b_i v_i \neq 0 \) (else we violate the linear independence of \( V \)) and if we set

\[
\mathbf{u} = \sum_{i=s+1}^{k} a_i \mathbf{u}_i + \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i + \sum_{i=1}^{s} c_i \mathbf{w}_i \quad \text{and} \quad \mathbf{v} = \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i \neq 0
\]

we have \( \mathbf{u} = -\mathbf{v} \neq 0 \). But \( -\mathbf{u} \in U \) and \( \mathbf{v} \in V \) and so \( \mathbf{v} \in U \cap V \). But \( \mathbf{v} \in \text{span} V \) and so \( \mathbf{v} \notin U \cap V \), a contradiction. Thus we deduce that \((U \setminus W) \cup W \cup (V \setminus W)\) are linearly independent. This yields \( \dim(U + V) = k + \ell - s \) where \( \dim U \cap V = s \). Hence

\[
\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V).
\]

7. Let \( V \) be a 3-dimensional subspace of \( \mathbb{R}^7 \). Show that the set of linear functions from \( V \) to \( \mathbb{R} \) is a vector space and determine its dimension (Hint: a linear function is determined by its action on a basis of the domain).

Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be basis for \( V \). We consider three linear functions \( V \to \mathbb{R} \) namely \( f_1, f_2, f_3 \) defined as follows. Any vector \( \mathbf{v} \) in \( V \) can be written \( \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \). Then define \( f_1(\mathbf{v}) = c_1, f_2(\mathbf{v}) = c_2 \) and \( f_3(\mathbf{v}) = c_3 \). The set of linear functions \( \text{span}\{f_1, f_2, f_3\} \) is a vector space of dimension 3. The three functions are linearly independent. Let \( i \in \{1, 2, 3\} \). Then \( f_i(\mathbf{v}_i) = 1 \neq 0 \) while \( f_j(\mathbf{v}_i) = 0 \) (since if we write \( \mathbf{v}_i \) as a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) the coefficient of \( \mathbf{v}_i \) will be 1 while the coefficient of the other basis vectors will be 0). But also any linear transformation \( f : V \to \mathbb{R} \) would be determined by its values on a basis so that if we have a linear transformation \( f : V \to \mathbb{R} \) with \( f(\mathbf{v}_i) = b_i \) for \( i \in \{1, 2, 3\} \), then \( f = b_1 f_1 + b_2 f_2 + b_3 f_3 \).

8. Consider the problem of 3D images with perspective. We are ‘projecting’ points in \( \mathbb{R}^3 \) onto a plane \( \pi : z = 1 \) where you consider your eye to be located at the origin (this is a typical problem in computer graphics). Given a point \( (x, y, z) \) we say its image on \( \pi \) is the intersection of the plane \( \pi \) with the line joining the point \( (x, y, z) \) and the origin. You might draw a picture to see what is going on. Now consider the image of a line \( L : \{x + sd : s \in \mathbb{R}\} \) on \( \pi \). What is its image (viewed as a sequence of points, one for each value of \( s \)) on the plane \( \pi \). Give a description for how the image is traced out on \( \pi \) as \( s \) goes from 0 to \( \infty \).

Let \( \mathbf{x} = (x_1, x_2, x_3)^T \) and \( \mathbf{d} = (d_1, d_2, d_3)^T \). The line \( L' \) joining the origin and \( \mathbf{x} + t \mathbf{d} \) (for a fixed value of \( t \)) is

\[
\{0 + s(\mathbf{x} + t \mathbf{d}) : s \in \mathbb{R}\} = \left\{ \begin{pmatrix} 0 \\ x_1 + td_1 \\ x_2 + td_2 \\ x_3 + td_3 \end{pmatrix} : s \in \mathbb{R}\right\}
\]

which hits the plane \( z = 1 \) when its \( z \) value is 1. This occurs for \( s = \frac{1}{x_3 + td_3} \). In particular, the projection is undefined for \( x_3 + td_3 = 0 \) since the line \( L' \) does not intersect the plane \( \pi \) in that case. We compute the points in \( \pi \) as a function of \( t \). We could argue that they lie on a line since the line \( L \) plus the origin determine a plane \( \pi' \) and the projections lie on the intersection \( \pi \cap \pi' \), which is a line. More algebraically an arbitrary point of the projection into \( \pi \) is

\[
\begin{pmatrix} x_1 + td_1 \\ x_2 + td_2 \\ x_3 + td_3 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} + \frac{td_1(x_3 - x_1) + td_2(x_2 - x_1)}{x_3 + td_3} \begin{pmatrix} 1 \\ 1 \\ x_3 + td_3 \end{pmatrix}
\]

We see that these points lie on a line by noting that

\[
\begin{align*}
\frac{td_1(x_3 - x_1)}{x_3 + td_3} &= \frac{d_1(x_3 - x_1)}{d_2(x_3 - x_2)} \\
\frac{td_2(x_2 - x_1)}{x_3 + td_3} &= \frac{d_2(x_3 - x_2)}{d_2(x_3 - x_2)}
\end{align*}
\]

which is independent of \( t \).
In addition we may compute

$$\lim_{t \to \infty} \begin{pmatrix} x_1 + td_1 \\ x_3 + td_3 \\ x_2 + td_2 \\ x_3 + td_3 \\ 1 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_3 \\ d_2 \\ d_3 \\ 1 \end{pmatrix}$$

where the direction \((d_1/d_3, d_2/d_3, 1)^T\) (a multiple of \(d\)) is the point at infinity visually speaking.