1. Determine bases for the following subspaces of $\mathbb{R}^3$.
   a) the line $x = 5t, y = -2t, z = t$ has basis
   $$\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$
   
   b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$ has the basis
   $$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \text{ or other choices}$$

2. Let
   $$A = \begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 2 & 6 & -6 & 0 \\ 0 & 3 & 7 & 2 & -9 & 7 \\ 0 & 2 & 4 & -4 & 3 \end{bmatrix}$$

   We have
   $$\begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 2 & 6 & -6 & 0 \\ 0 & 3 & 7 & 2 & -9 & 7 \\ 0 & 2 & 4 & -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   A basis for the column space of $A$ is
   $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ -9 \\ -4 \end{bmatrix}$$

   and a basis for the row space of $A$ is
   $$(0, 1, 1, 2, -3, 1)^T, (0, 0, -2, 2, 0, -2)^T, (0, 0, 0, 0, 2, 1)^T$$

   A basis for the nullspace of $A$ is
   $$\begin{bmatrix} 0 \\ -3/2 \\ -1 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   In all cases, other bases are possible.

   such questions appear on your practice for Midterm 2 although in those questions the reduction to staircase pattern has already occurred, saving you some computation.

3. Let $V = \{b : Ax = b \text{ is consistent}\}$. Then if $u, v \in V$, we have That there exists a $y$ with $Ay = u$ and there exists a $z$ with $Az = v$. But then $A(y + z) = Ay + Az = u + v$ and hence $u + v \in V$. Similarly $A(ky) = kAy = ku$. Hence $ku \in V$ as well and so $V$ is a vector space. An
easier proof is that \( \mathbf{b} \in V \) if and only if \( \mathbf{b} \in \text{colsp}(A) \) and we already know that the \( \text{colsp}(A) \) is a vector space (it is the span of the columns of \( A \)).

\[
\begin{bmatrix}
2 & 3 & 1 & b_1 \\
4 & 3 & 3 & b_2 \\
1 & 3 & 0 & b_3 \\
2 & 0 & 2 & b_4
\end{bmatrix} \rightarrow
\begin{bmatrix}
2 & 3 & 1 & b_1 \\
0 & -3 & 1 & b_2 - 2b_1 \\
0 & 0 & 0 & -3/2b_1 + 1/2b_2 + b_3 \\
0 & 0 & 0 & b_1 - b_2 + b_4
\end{bmatrix}
\]

The system is consistent if \(-b_1 + 1/2b_2 + b_3 = 0\) and \(b_1 - b_2 + b_4 = 0\). We now solve this system in \(b_1, b_2, b_3, b_4\) to obtain

\[
\begin{bmatrix}
-3/2 & 1/2 & 1 & 0 \\
1 & -1 & 0 & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
-3/2 & 1/2 & 1 & 0 \\
0 & -2/3 & 2/3 & 1
\end{bmatrix}
\]

and so the basis is

\[
\begin{bmatrix}
1/2 \\
3/2 \\
0 \\
1
\end{bmatrix} \quad \text{or other choices}
\]

I mention other choices and in this case it would be slightly easier to identify \(b_3, b_4\) as the pivot variables and \(b_1, b_2\) as the free variables.

Again, a different and easier solution is to identify the set of possible \( \mathbf{b} \) as \( \text{colsp}(A) \) and then, by Gaussian Elimination, a basis is

\[
\begin{bmatrix}
2 \\
4 \\
1 \\
2
\end{bmatrix} \begin{bmatrix}
3 \\
3 \\
3 \\
0
\end{bmatrix}
\]

4. We say two \( n \times n \) matrices \( A, B \) are similar if there is an invertible matrix \( M \) with \( A = MBM^{-1} \). We have shown that \( A, B \) being similar implies \( \det(A - \lambda I) = \det(B - \lambda I) \). Assume \( A \) has \( k \leq n \) linearly independent eigenvectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) of an eigenvalue 2.

a) Given \( k \leq n \) linearly independent eigenvectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) of \( A \) of eigenvalue 2 in \( \mathbb{R}^n \) we can always extend to an invertible \( n \times n \) matrix \( M \) where the first \( k \) columns of \( M \) are \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \). There are several possible proofs of this. In general we are just extending a linearly independent set of vectors in a vector space to a basis. One step in the process is to seek a vector \( \mathbf{u} \not\in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \) and then \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}\} \) is a linearly independent set of vectors (any dependency involving \( \mathbf{u} \) can be rewritten as an expression for \( \mathbf{u} \) as a linear combination of vectors from \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \). Continue until we have a basis.

If we follow the ideas in 5) we can create an \( n \times (k + n) \) matrix \( C = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n] \) and then \( \text{colsp}(C) = \text{colsp}([\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n] = \mathbb{R}^n \). We may use Gaussian elimination to select a column basis for \( C \) from the columns of \( C \) by using the pivot columns but because the initial \( k \) columns are linearly independent the initial \( k \) columns will be selected as part of a basis of \( n \) vectors for the \( \text{colsp}(C) \). These \( n \) columns can be chosen as the \( n \) columns of \( M \).

b) We compute

\[
M^{-1}AM = M^{-1}[2\mathbf{u}_1 2\mathbf{u}_2 \cdots 2\mathbf{u}_k \mathbf{B}]
\]

\[
\begin{bmatrix}
2 & 0 & \cdots & 0 \\
0 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & 2
\end{bmatrix}
\]

\[
0's
\]
Now we apply $\det(A - \lambda I) = \det(B - \lambda I)$ and compute $\det(A - \lambda I)$ as $\det(M^{-1}AM - \lambda I) = (2 - \lambda)^k p(\lambda)$ where $p(\lambda)$ is a polynomial in $\lambda$ of degree $n - k$. Thus the multiplicity of 2 as a root of $\det(A - \lambda I)$ is at least $k$.

In summary, the dimension of the eigenspace for $\lambda_i$ is at most the multiplicity of $\lambda_i$ as a root in the characteristic equation $\det(A - \lambda I)$.

5. Let $A$ be an $n \times n$ matrix with various eigenvalues including $\lambda$ and $\mu$. Let $L, M$ be the eigenspaces associated with eigenvalues $\lambda, \mu$ respectively. Let $\{u_1, u_2, \ldots, u_p\}$ be a basis for $L$ and let $\{v_1, v_2, \ldots, v_q\}$ be a basis for $M$. Assume that $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}$ are linearly dependent, namely we can find $a_1, a_2, \ldots, a_{p+q}$ with $a_1 u_1 + a_2 u_2 + \cdots + a_p u_p + a_{p+1} v_1 + a_{p+2} v_2 + \cdots + a_{p+q} v_q = 0$. Let $x = a_1 u_1 + a_2 u_2 + \cdots + a_p u_p$ and $y = a_{p+1} v_1 + a_{p+2} v_2 + \cdots + a_{p+q} v_q$ so that $Ax = \lambda x$ and $Ay = \mu y$ with $x + y = 0$.

Case 1. $x \neq 0$ and so $y \neq 0$.

We have $\mathbf{0} = A \mathbf{0} = A(x + y) = \lambda x + \mu y$. But now one of $\lambda, \mu \neq 0$ (they are not equal), say $\lambda \neq 0$ and so $x = (-\mu/\lambda)y$. But then $\lambda x = Ax = (-\mu/\lambda)Ay = (\mu^2/\lambda)y$ and then this forces $\lambda(-\mu/\lambda) = (-\mu^2/\lambda)$ which has $\lambda = \mu$ as the only conclusion, a contradiction.

Case 2. $x = 0$ and so $y = 0$.

With $x = 0$, we have $\mathbf{0} = a_1 u_1 + a_2 u_2 + \cdots + a_p u_p$ and so by the linear independence of $u_1, u_2, \ldots, u_p$, means $a_1 = a_2 = \cdots = a_p = 0$ and similarly with $y = 0$, we have $\mathbf{0} = a_{p+1} v_1 + a_{p+2} v_2 + \cdots + a_{p+q} v_q$ and the linear independence of $v_1, v_2, \ldots, v_q$ forces $a_{p+1} = a_{p+2} = \cdots = a_{p+q} = 0$.

Thus we have shown that Case 1 cannot occur and Case 2 yields linear independence.

What if there were three different eigenvalues and three bases for the eigenspaces? It suffices to consider $u, v, w$ eigenvectors with $Au = \lambda u$, $Av = \mu v$ and $Aw = \rho w$ where $\lambda \neq \mu \neq \rho$. Assume $au + bv + cw = 0$. Then $A(au + bv + cw) = A\mathbf{0} = \mathbf{0}$. But $A(au + bv + cw) = a\lambda u + b\mu v + c\rho w = \mathbf{0}$. Now $au + bv + cw = 0$ yields $a\lambda u + b\mu v + c\rho w = 0$ and so we deduce $b(\mu - \lambda)v + c(\rho - \lambda)w = 0$. But we already have $v, w$ are linearly independent and so $b(\mu - \lambda) = 0$ and $c(\rho - \lambda) = 0$. Given that $\lambda \neq \mu \neq \rho$, we deduce that $b = c = 0$. From this we deduce that $a = 0$ and hence $u, v, w$ are linearly independent. This is the important part in showing that if we have three different eigenvalues and three bases for the eigenspaces then the union of the bases are linearly independent. You can probably see this also works for 4 or more different eigenvalues.

6. We can show that $f$ is a linear transformation by checking $f(A + B) = (A + B)^T = A^T + B^T = f(A) + f(B)$,

$$f(kA) = (kA)^T = kA^T = kf(A).$$

It is straightforward to note that (non-zero) symmetric matrices correspond to eigenvectors of eigenvalue 1 for $f$ and similarly (non-zero) skew-symmetric matrices correspond to eigenvectors of eigenvalue -1 for $f$.

Now we can show that $\dim(\text{symmetric matrices}) = 6$ with the basis

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
Now we can show that \( \dim(\text{skew-symmetric matrices}) = 3 \) with the basis

\[
S_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},
\]

Now combining the basis for the eigenspace of eigenvalue 1 of 6 vectors \( \{S_1, S_2, S_3, S_4, S_5, S_6\} \) and the basis for the eigenspace of eigenvalue -1 of 3 vectors \( \{T_1, T_2, T_3\} \), we obtain a linearly independent set of 9 vectors by the question above. Now \( \dim(M_{3\times 3}) = 9 \) (we can form a basis of the 9 \( 3 \times 3 \) matrices each with a single 1) and so our 9 vectors \( \{S_1, S_2, S_3, S_4, S_5, S_6, T_1, T_2, T_3\} \) must be a basis for \( M_{3\times 3} \). Now any matrix in \( A \in M_{3\times 3} \) can be written as a linear combination of the 9 vectors. Thus \( A \) can be written as the sum of a linear combination of the first 6 (which yields a symmetric matrix) and a linear combination of the last 3 (which yields a skew-symmetric matrix).

There is a much easier way to see that any matrix \( A \) is a sum of a symmetric matrix and a skew-symmetric matrix by writing \( A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \) where we note that \( \frac{1}{2}(A + A^T) \) is symmetric and \( \frac{1}{2}(A - A^T) \) is skew-symmetric. But I think the argument using dimensions is prettier for this course.

7. Write a proof of the result quoted in class namely that for every matrix \( A \), the rank(\( A \)) is the maximum \( k \) such that \( A \) has a \( k \times k \) submatrix which is invertible. Recall that a submatrix of \( A \) is obtained by deleting rows and columns (such a minor in our definition of determinant) or vice versa by selecting the matrix whose entries are in specified rows and columns of \( A \). The set of rows and columns selected can be quite different.

Assume rank(\( A \)) = \( k \). Then we show that there is a \( k \times k \) submatrix of \( A \) which has rank \( k \). To do so, note that rank(\( A \)) = dim(colsp(\( A \))) and so there are \( k \) columns of \( A \) which are linearly independent. Let the matrix formed by these \( k \) columns be denoted \( B \). Now rank(\( B \)) = \( k \) since colsp(\( B \)) \( \subset \) colsp(\( A \)) and dim(colsp(\( B \))) = \( k \). Now repeat on \( B^T \). We have rank(\( B^T \)) = rank(\( B \)) and so there are \( k \) linearly independent columns in \( B^T \) which of course corresponds to \( k \) linearly independent rows in \( B \). Let \( C \) be the submatrix of \( B \) formed by these \( k \) rows. We have rowsp(\( C \)) \( \subseteq \) rowsp(\( B \)) and rank(\( C \)) = \( k \). Now \( C \) is a \( k \times k \) submatrix of \( A \) of rank \( k \).

Assume \( A \) has a \( p \times q \) submatrix \( D \), then rank(\( A \)) \( \geq \) rank(\( D \)). This is because if \( D \) has \( t \) linearly independent columns then by extension \( A \) has \( t \) linearly independent columns. Thus if \( A \) has a \( k \times k \) submatrix of rank \( k \), then rank(\( A \)) \( \geq k \).

Together, these two observations prove the desired result.

8.

\[
\det \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ r_1 & r_2 & r_3 & \ldots & r_{n-1} & r_n \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_{n-1}^2 & r_n^2 \\ r_1^3 & r_2^3 & r_3^3 & \ldots & r_{n-1}^3 & r_n^3 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & r_3^{n-1} & \ldots & r_{n-1}^{n-1} & r_n^{n-1} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ r_1 & r_2 - r_1 & r_3 - r_1 & \ldots & r_{n-1} - r_1 & r_n - r_1 \\ r_1^2 & r_2^2 - r_1^2 & r_3^2 - r_1^2 & \ldots & r_{n-1}^2 - r_1^2 & r_n^2 - r_1^2 \\ r_1^3 & r_2^3 - r_1^3 & r_3^3 - r_1^3 & \ldots & r_{n-1}^3 - r_1^3 & r_n^3 - r_1^3 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} - r_1^{n-1} & r_3^{n-1} - r_1^{n-1} & \ldots & r_{n-1}^{n-1} - r_1^{n-1} & r_n^{n-1} - r_1^{n-1} \end{bmatrix}
\]
(using column operations of subtracting column 1 from the other columns in turn)

\[
\prod_{i=2}^{n} (r_i - r_1) \det \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
(r_1 + 1) & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \sum_{i=0}^{n-2} r_{2i} r_{1}^{-2-i} & \sum_{i=0}^{n-2} r_{3i} r_{1}^{-2-i} & \cdots & \sum_{i=0}^{n-2} r_{ni} r_{1}^{-2-i}
\end{bmatrix}
\]

(pulling out common factors of \((r_i - r_1)\) from each column)

\[
\prod_{i=2}^{n} (r_i - r_1) \det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
r_2 & r_3 & \cdots & r_{n-1} & r_n \\
r_2^2 & r_2 r_3 & \cdots & r_{n-2} & r_{n-1} \\
r_2^3 & r_2^2 r_3 & \cdots & r_{n-1} r_{n-2} & r_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_2^{n-2} & r_2^{n-3} & \cdots & r_{n-2} r_{n-3} & r_{n-2}
\end{bmatrix}
\]

(using expansion about the first row and then subtracting \(r_1\) times the second last row from the last row and also subtracting \(r_1\) times the third last row from the second last row etc always using the identity

\[
\sum_{i=0}^{t} r_i^{j} r_1^{t-i} - r_1 \sum_{i=0}^{t-1} r_i^{j} r_1^{t-1-i} = r_j^t
\]

to simplify)

\[
\prod_{i=2}^{n} (r_i - r_1) \cdot \prod_{2 \leq i < j \leq n} (r_j - r_i) = \prod_{1 \leq i < j \leq n} (r_j - r_i)
\]

using induction for the \((n - 1) \times (n - 1)\) submatrix.