1. Let $A$ be an $m \times n$ matrix of rank 1. Show that there exist non zero vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ so that $A = xy^T$. (Hint: Try a simple case. Also compute $xy^T$ for some simple choices for $x,y$.)

2. Let $\{u_1,u_2,u_3\}$ be a basis for a vector space $V$. Then if we define $v_1 = u_1 + 2u_3$, $v_2 = u_1 + 2u_2 + 3u_3$, $v_3 = u_2 - u_3$, show that $\{v_1,v_2,v_3\}$ forms a basis for $V$.

3. Assume we have a polynomial that we can factor into linear factors. We refer to the multiplicity of a root in the polynomial. Thus $(\lambda - 2)^3(\lambda - 1)$ has 2 as a root with multiplicity 3, as well as 1 as a root with multiplicity 1. We are always interested in the characteristic polynomial $\det(A - \lambda I)$. Assume $A$ is $n \times n$ and that the characteristic polynomial factors into linear factors (there may be repeats). Show that $\det(A)$ is the product of the eigenvalues, according to their multiplicities and that $\text{tr}(A)$ is the sum of the eigenvalues, according to their multiplicities. For the latter problem you should at least do this in the $3 \times 3$ case.

4. Let $A$ be a $4 \times 4$ diagonalizable matrix with $\det(A - \lambda I) = (\lambda - 2)^3(\lambda - 4)$. Determine rank($A$). Determine rank($A - 2I$). Determine rank($A - 4I$).

5. (from a test)

Let $w_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $w_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\text{NOTE:} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation satisfying

$f(w_1) = w_2 - w_3$, \hspace{5mm} $f(w_2) = -w_2 + w_3$, \hspace{5mm} $f(w_3) = w_1 + w_2 + w_3$.

a) Give the matrix representation of $f$ with respect to the basis $\{w_1,w_2,w_3\}$.

b) Give the matrix representation of $f$ where the input $x$, is written with respect to the basis $\{w_1,w_2,w_3\}$ and the output $f(x)$ is written with respect to the basis $\{e_1,e_2,e_3\}$ (the standard basis).

c) Is $w_1$ in the range of $f$?

6. (from a test)

Let $z_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $z_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $z_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\text{NOTE:} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation satisfying

$T(z_1) = 2z_2$, \hspace{5mm} $T(z_2) = 2z_2$, \hspace{5mm} $T(z_3) = z_1 + z_2$.

a) Give the matrix representation of $T$ with respect to the basis $\{z_1,z_2,z_3\}$.

b) Give the matrix representation of $T$ with respect to the basis $\{e_1,e_2,e_3\}$ (the standard basis). Give the explicit matrix with integer entries.
c) Give the matrix representing $T^2$ with respect to the basis $\{z_1, z_2, z_3\}$. What is the rank of the matrix representing $T^2$ with respect to the standard basis $\{e_1, e_2, e_3\}$?

7. Given a polynomial $p(x) = a_0 + a_1 x + \cdots + a_k x^k$, we can also consider a polynomial in a (square) matrix $A$ by defining $p(A) = a_0 I + a_1 A + \cdots + a_k A^k$. Show that if $x$ is a eigenvector of $A$ of eigenvalue $\mu$, then $x$ is an eigenvector of $p(A)$ with eigenvalue $p(\mu)$. Use this idea to show that for an $n \times n$ matrix $J$, $\det(r \cdot J + k \cdot I) = (nr + k)^{n-1}$. Hint: Question 8 from Assignment 5.

8. Let $U$ and $V$ be two 5-dimensional subspaces of $\mathbb{R}^9$. Show that there is a nonzero vector in $U \cap V$, the intersection of $U$ and $V$. Using Gaussian elimination should make this reasonable. You may find it helpful to consider the case of two 2-dimensional subspaces of $\mathbb{R}^3$ first but you won’t be able to use ideas of lines and planes in $\mathbb{R}^9$. 