Math 223 Assignment #4 due Friday October 3 in class. Midterm is scheduled for Monday October 6. A practice midterm is posted.

1. Compute

i) \( \det \begin{bmatrix} 3 & -5 \\ -6 & 10 \end{bmatrix} \),

ii) \( \det \begin{bmatrix} 3 & 6 & 2 \\ 4 & 7 & \pi \\ 1 & 2 & 0 \end{bmatrix} \),

iii) \( \det \begin{bmatrix} 0 & 3 & 101 \\ 0 & e & 97 \\ 0 & 1 & 98 \end{bmatrix} \),

iv) \( \det \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \)

2. Use Cramer’s rule to find the inverse of the following (when it exists)

\[ A = \begin{bmatrix} x & 1 & 1 \\ 1 & 1 & x \\ x & 2 & 1 \end{bmatrix} \]

3. On the first midterm you will be expected to compute the eigenvalues and associated eigenvectors for a 3 \( \times \) 3 matrix in much the same way you did question 2 on assignment 2. Let

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix} \]

a) Find the eigenvectors of eigenvalue 2. (this should be just Gaussian Elimination)

b) Compute \( \det(A - \lambda I) \) (a cubic polynomial in \( \lambda \)) by the expansion method (using Gaussian elimination may split into cases; don’t use it). A check on your work is that \((\lambda - 2)\) should be a factor of the polynomial (why? because 2 is a root).

c) Factorize \( \det(A - \lambda I) \) and determine all eigenvalues and for each eigenvalue, describe the associated set of eigenvectors.

4. Let \( A \) be an \( n \times n \) matrix with an eigenvector \( \mathbf{v} \) of eigenvalue \( \lambda \). Assume we can obtain an invertible matrix \( M \) which has \( \mathbf{v} \) as its first column. Compute the first column of \( B = M^{-1}AM \). Deducce that \( B \) has an eigenvalue \( \lambda \).

5. Assume that \( A = MBM^{-1} \). Show that \( \det(A - \lambda I) = \det(B - \lambda I) \). Product rule?

6. Assume \( A \) is a 3 \( \times \) 3 matrix, and \( M \) is an invertible matrix with \( A = MDM^{-1} \), where \( D \) is the diagonal matrix

\[ D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \]

Show that \((A - 2I)(A - 3I)(A - 4I) = 0.\)

7. Let \( A = (a_{ij}) \) be a matrix with integral entries such that the diagonal entries are all odd \( (a_{ii} \) is odd) and all off diagonal entries are even \( (a_{ij} \) is even for \( i \neq j \)). Show that \( A \) has \( \det(A) \neq 0, \) i.e. \( A \) is invertible.