1. Let $A = \begin{bmatrix} x & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ x & y \end{bmatrix}$. Determine all $x, y$ so that $AB = BA$.

2. Find a $2 \times 2$ matrix $A$, no entry of which is 0, with $A^2 = A$. Note that your first guesses $A = I$ or $A = 0$ have 0 entries.

3. Gavin has a sum of $s$ dollars in $t$ bills (not treasury bills!) The bills are either $3$ or $5$. Express the number $x$ of $3$ bills and the number $y$ of $5$ bills in terms of $s$ and $t$.

4. Assume you are given a pair of matrices $A, B$ which satisfy $AB = BA$. Show that if we set $C = A^2 + 2A$ and $D = B^3 + 5I$, then $CD = DC$. Generalize this, namely find a property so that for matrices $C, D$ with that certain property, then $CD = DC$.

5. Let $R(\theta)$ denote the matrix of the transformation which rotates the plane by $\theta$ counterclockwise around the origin. Explain in terms of transformations why $R(\theta)R(\phi) = R(\theta + \phi)$. Show how you can use this to derive the formulas for $\cos(\theta + \phi), \sin(\theta + \phi)$ in terms of $\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)$.

6. Find a matrix $A$ with $A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. (We can rephrase this as finding a matrix $A$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of eigenvalue 3 and eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ of eigenvalue 2).

7. a) Assume $A, B$ are $2 \times 2$ invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.
   b) Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then define $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, where $A^T$ is called the transpose of $A$. Show that $(AB)^T = B^TA^T$ for any $2 \times 2$ matrices $A, B$. Explain how to justify this by realizing that $A^T$ corresponds interchanging the roles of rows and columns in $A$.

8. Define $\text{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$ (read trace for ‘tr’).
   a) Using $A^* = (a + d)I - A$ verify $AA^* = (ad - bc)I$ and verify the Cayley-Hamilton Theorem (at least for $2 \times 2$ matrices):
      $A^2 - \text{tr}(A)A + \text{det}(A)I = 0$
   (i.e. $A$ acts as a ‘root’ to the quadratic $\text{det}(A - \lambda I)$ when interpreted as a matrix polynomial).
   b) Determine conditions on $\text{tr}(A)$ and $\text{det}(A)$ to ensure that $A^2 = A$ but $A \neq I, O$. Hint: Consider the two cases $A \neq kI$ for any $k$ and $A = kI$ for some $k$.
   c) Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $A^2 = A$ and $A \neq I, O$, determine formulas for $c, d$ in terms of $a, b$.

9. Consider two nonzero vectors $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{y} = \begin{bmatrix} c \\ d \end{bmatrix}$. We would like to establish a simple condition on $a, b, c, d$ that determines whether changing from direction $\mathbf{x}$ to direction $\mathbf{y}$ corresponds to turning left or right. One possible way is to note that there is a rotation matrix $R(\theta)$ so that $\mathbf{y} = R(\theta)\mathbf{x}$ and $0 \leq \theta < 2\pi$. Use our knowledge of rotation matrices to establish a simple condition on $a, b, c, d$ so that the angle $\theta$ satisfies $0 < \theta < \pi$. You can even assume $a, b, c, d$ are all positive and nonzero, if that assists you.