Be sure this exam has 3 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 2009

MATH 223: Linear Algebra

Instructor: Dr. R. Anstee, section 102

Special Instructions: No Aids. No calculators or cellphones. time: 3 hours
You must show your work and explain your answers.

1. (15 marks) Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 & 1 \\ 2 & 1 & 4 & 5 & 2 & 2 \\ 3 & -1 & 1 & 5 & 3 & 3 \\ 1 & -1 & -1 & 1 & 2 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ 15 \\ 15 \\ 5 \end{bmatrix}$$

There is an invertible matrix M so that

$$MA = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M\mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 2 \\ 0 \end{bmatrix}$$

a) (2 marks) What is rank(A)?

b) (1 marks) What is rank(M)?

c) (4 marks) Give a vector parametric form for the set of solutions to $A\mathbf{x} = \mathbf{b}$.

d) (6 marks) Give a basis for the row space of A. Give a basis for the column space of A. Give a basis for the null space of A.

e) (2 marks) How many columns would be required to add to A so that the resulting matrix has full rank; namely rank 4. Would adding $\mathbf{b}$ to A be any help?

2. (15 marks) Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Determine an orthonormal basis of eigenvectors and hence an orthogonal matrix Q and a diagonal matrix D so that $AQ = QD$. You may find it useful to know that 0 is an eigenvalue of A.

3. (5 marks) Determine an equation of the plane through the origin whose set of vectors is the vector space spanned by $(1, 2, 3)^T$ and $(2, 3, 4)^T$.

4. (10 marks) a) (3 marks) Let Z denote the 3×3 matrix of all zeroes. Give an orthogonal diagonalizing matrix for Z.

   b) (3 marks) Compute $||\frac{4}{223}(3,2,1)|^T|$.

   c) (4 marks) Compute the matrix representing the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ (with respect to the standard basis) with T defined as

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2\begin{bmatrix} x \\ y \end{bmatrix} - (x+2y)\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

5. (10 marks) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for $\mathbf{R}^2$ and let T be the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T(\mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2, \qquad T(\mathbf{v}_2) = \mathbf{v}_2.$$

   a) (5 marks) Find the matrix representing T with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

   b) (5 marks) Find the matrix representing T with respect to the standard basis when we are given that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

6. (5 marks) Let U, V be bases for $\mathbf{R}^3$ and let E be the standard basis. Let A be the change of basis matrix going from U coordinates to E coordinates, let B be the change of basis matrix going from V coordinates to E coordinates, and let C be the matrix going from U coordinates to V coordinates. Also let D be the matrix corresponding to the linear transformation T written with respect to the basis U. Please give a simple interpretation (as simple as possible) for the matrix F, given below:

$$F = BCDA^{-1}$$

7. (10 marks)

$$\text{Let } S = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ -2 \\ -5 \end{bmatrix}\right\}.$$

   a) (5 marks) Find an orthogonal basis for S.

   b) (5 marks) Project the vector $(10,10,0,0)^T$ onto the vector space S. (The numbers should work out nicely). You can check your work by subtracting the projection from $(10,10,0,0)^T$ and seeing if the resulting vector is in $S^\perp$. Indicate (without having to solve) the system of equations you would have to solve in order to find the 'least squares' estimate for $(x,y,z)^T$ in the 'equation'

$$\begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & -4 \\ 2 & 3 & -2 \\ 1 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix}.$$

8. (10 marks) Assume A is a symmetric 3×3 matrix with $\det(A - \lambda I) = -(\lambda - 2)(\lambda - 3)^2$. Assume A has $(1, 1, 2)^T$ as an eigenvector of eigenvalue 2. Give a vector parametric form for the eigenspace of eigenvalue 3.

9. (10 marks) Assume A is a 3×3 matrix with eigenvalues $1, \frac{1}{2}, -\frac{1}{2}$ with associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let a, b, c be given and let $\mathbf{x}_0 = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$. Then we may generate a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ using the recursion $\mathbf{x}_{i+1} = A\mathbf{x}_i$ for $i \geq 0$. Show that

$$\lim_{n \to \infty} \mathbf{x}_n = a\mathbf{v}_1.$$

10. (10 marks) Let Q be an orthogonal matrix. Given two vectors $\mathbf{x}, \mathbf{y}$, we can define the distance between $\mathbf{x}$ and $\mathbf{y}$ as $||\mathbf{x} - \mathbf{y}||$. Show that

$$||\mathbf{x} - \mathbf{y}|| = ||Q\mathbf{x} - Q\mathbf{y}||$$

(we may interpret this as saying that the linear transformation corresponding to an orthogonal matrix will preserve distances).

100 Total marks