1. (15 marks) Consider the matrix equation $Ax = b$ with

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 & 1 \\ 2 & 1 & 4 & 5 & 2 & 2 \\ 3 & -1 & 1 & 5 & 3 & 3 \\ 1 & -1 & -1 & 1 & 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 15 \\ 15 \\ 5 \end{bmatrix}$$

There is an invertible matrix $M$ so that

$$MA = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Mb = \begin{bmatrix} 6 \\ 3 \\ 2 \\ 0 \end{bmatrix}$$

a) (2 marks) What is rank($A$)?

b) (1 marks) What is rank($M$)?

c) (4 marks) Give a vector parametric form for the set of solutions to $Ax = b$.

d) (6 marks) Give a basis for the row space of $A$. Give a basis for the column space of $A$. Give a basis for the null space of $A$.

e) (2 marks) How many columns would be required to add to $A$ so that the resulting matrix has full rank; namely rank 4. Would adding $b$ to $A$ be any help?

2. (15 marks) Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Determine an orthonormal basis of eigenvectors and hence an orthogonal matrix $Q$ and a diagonal matrix $D$ so that $AQ = QD$. You may find it useful to know that 0 is an eigenvalue of $A$.

3. (5 marks) Determine an equation of the plane through the origin whose set of vectors is the vector space spanned by $(1, 2, 3)^T$ and $(2, 3, 4)^T$. 
4. (10 marks) a) (3 marks) Let $Z$ denote the $3 \times 3$ matrix of all zeroes. Give an orthogonal diagonalizing matrix for $Z$.

b) (3 marks) Compute $\|\frac{1}{\sqrt{23}} (3, 2, 1)^T\|$.

c) (4 marks) Compute the matrix representing the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (with respect to the standard basis) with $T$ defined as

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = 2 \begin{bmatrix} x \\ y \end{bmatrix} - (x + 2y) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

5. (10 marks) Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$ and let $T$ be the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$T(v_1) = 2v_1 + v_2, \quad T(v_2) = v_2.$$

a) (5 marks) Find the matrix representing $T$ with respect to the basis $\{v_1, v_2\}$.

b) (5 marks) Find the matrix representing $T$ with respect to the standard basis when we are given that

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

6. (5 marks) Let $U, V$ be bases for $\mathbb{R}^3$ and let $E$ be the standard basis. Let $A$ be the change of basis matrix going from $U$ coordinates to $E$ coordinates, let $B$ be the change of basis matrix going from $V$ coordinates to $E$ coordinates, and let $C$ be the matrix going from $U$ coordinates to $V$ coordinates. Also let $D$ be the matrix corresponding to the linear transformation $T$ written with respect to the basis $U$. Please give a simple interpretation (as simple as possible) for the matrix $F$, given below:

$$F = BCDA^{-1}. $$

7. (10 marks)

Let $S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ -2 \\ -5 \end{bmatrix} \right\}.$$

a) (5 marks) Find an orthogonal basis for $S$.

b) (5 marks) Project the vector $(10, 10, 0, 0)^T$ onto the vector space $S$. (The numbers should work out nicely). You can check your work by subtracting the projection from $(10, 10, 0, 0)^T$ and seeing if the resulting vector is in $S^\perp$. Indicate (without having to solve) the system of equations you would have to solve in order to find the ‘least squares’ estimate for $(x, y, z)^T$ in the ‘equation’

$$\begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & -4 \\ 2 & 3 & -2 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix}. $$
8. (10 marks) Assume $A$ is a symmetric $3 \times 3$ matrix with $\det(A - \lambda I) = -(\lambda - 2)(\lambda - 3)^2$. Assume $A$ has $(1, 1, 2)^T$ as an eigenvector of eigenvalue 2. Give a vector parametric form for the eigenspace of eigenvalue 3.

9. (10 marks) Assume $A$ is a $3 \times 3$ matrix with eigenvalues $1, \frac{1}{2}, -\frac{1}{2}$ with associated eigenvectors $v_1, v_2, v_3$. Let $a, b, c$ be given and let $x_0 = av_1 + bv_2 + cv_3$. Then we may generate a sequence $x_0, x_1, x_2, \ldots$ using the recursion $x_{i+1} = Ax_i$ for $i \geq 0$. Show that

$$\lim_{n \to \infty} x_n = av_1.$$ 

10. (10 marks) Let $Q$ be an orthogonal matrix. Given two vectors $x, y$, we can define the distance between $x$ and $y$ as $||x - y||$. Show that

$$||x - y|| = ||Qx - Qy||$$

(we may interpret this as saying that the linear transformation corresponding to an orthogonal matrix will preserve distances).

100 Total marks