MATH 184 Runners problem

Intermediate Value Theorem


We consider the situation of a runner who finished a 12km race in 48 minutes. This is on average 4 minutes per km. The runner is interested in running in an upcoming 10km race and wonders if he can do it in 40 minutes.

Question 1. Did he complete a 10 km race in exactly 40 minutes in a 10 km segment of his 12 km race?

Question 2. Did he complete a 6 km race in exactly 24 minutes in a 6 km segment of his 12 km race?

The answer to Question 2 is Yes and the answer to Question 1 is no (or in particular, not necessarily).

For Question 2 we define a function \( f(x) \) that considers the various 6 km segments that could be considered. We define \( f(x) = \) time required to run from \( x \) km to \( x + 6 \) km segment of 12 km race. Thus \( f \) has domain \([0, 6]\). We observe that \( f \) is continuous and so can apply the Intermediate Value Theorem.

We consider \( f(0) \) and \( f(6) \). We note that \( f(0) + f(6) = 48 \) since the two 6 km segments covers the entire 12 km race. Now if \( f(0) = 24 \) we are done and have Yes answer. If \( f(0) > 24 \), then \( f(6) < 24 \) using \( f(0) + f(6) = 48 \). But now we may appeal to the Intermediate Value Theorem to deduce there is some \( c \in [0, 6] \) with \( f(c) = 24 \). Thus we have run the 6 km (from \( c \) km to \( c + 6 \) km) in 24 minutes answering Yes to Question 2.

For Question 1 the following (reasonable) running pattern demonstrates a case where the answer in No. Run the first km in 3 minutes, the next 10 kms in 42 minutes at a constant pace and the last km in 3 minutes. As for question 1, we may define a continuous function \( g(x) = \) time required to run from \( x \) km to \( x + 10 \) km segment of 12 km race. Thus \( g \) has domain \([0, 2]\). But for our given running pattern, \( g(x) > 24 \) for all \( x \) in the domain. e.g. \( g(0) = 3 + 9 \times 4.2 = 40.8 \) minutes. This yields No to Question 1 (at least we cannot conclude Yes).