Forbidden Configurations: Boundary Cases

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Consider the following family of subsets of \( \{1, 2, 3, 4\} \):
\[
A = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}
\]
The incidence matrix \( A \) of the family \( A \) of subsets of \( \{1, 2, 3, 4\} \) is:
\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

**Definition** We say that a matrix \( A \) is *simple* if it is a \((0,1)\)-matrix with no repeated columns.

**Definition** We define \( \|A\| \) to be the number of columns in \( A \).
\[
\|A\| = 6 = |A|
\]
**Definition** Given a matrix $F$, we say that $A$ has $F$ as a configuration if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$
**Definition**  Given a matrix $F$, we say that $A$ has $F$ as a configuration if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

We consider the property of forbidding a configuration $F$ in $A$.

**Definition**  Let

$$forb(m, F) = \max \{ ||A|| : A \text{ $m$-rowed simple, no configuration $F$} \}$$
A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$, e.g:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
We conjecture that our product constructions with the three building blocks \( \{I, I^c, T\} \) determine the asymptotically best constructions.
**Definition** Given two matrices $A, B$, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of $A$ on top of a column of $B$ in all possible ways. (A, Griggs, Sali 97)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \times 
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.
Examples

\[
[01] \times [01] = K_2 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

\( I_{m/2} \times I_{m/2} \) is vertex-edge incidence matrix of \( K_{m/2,m/2} \)
The Conjecture

We conjecture that our product constructions with the three building blocks \( \{I, I^c, T\} \) determine the asymptotically best constructions.

**Definition** Let \( F \) be given. Let \( x(F) \) denote the largest \( p \) such that there is a \( p \)-fold product which does not contain \( F \) as a configuration where the \( p \)-fold product is \( A_1 \times A_2 \times \cdots \times A_p \) where each \( A_i \in \{l_{m/p}, l_{m/p}^c, T_{m/p}\} \).

**Conjecture** (A, Sali 05) \( \text{forb}(m, F) \) is \( \Theta(m^{x(F)}) \).
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**Conjecture** (A, Sali 05) \( \text{forb}(m, F) \) is \( \Theta(m^{x(F)}) \).

The conjecture has been verified for \( k \times \ell \ F \) where \( k = 2 \) (A, Griggs, Sali 97) and \( k = 3 \) (A, Sali 05) and \( \ell = 2 \) (A, Keevash 06) and for \( k \)-rowed \( F \) with bounds \( \Theta(m^{k-1}) \) or \( \Theta(m^k) \) (A, Fleming 10) plus other cases.
Definition Let $F$ be a $k$-rowed configuration and let $\alpha$ be a $k$-rowed column vector. Define $[F|\alpha]$ to be the concatenation of $F$ and $\alpha$.

Definition Let $F$ be a $k$-rowed configuration. We say that $F$ is a boundary case if for every $k$-rowed column $\alpha$ which is either not present in $F$ or present just once in $F$, then $forb(m,[F|\alpha])$ is $\Omega(m \cdot forb(m,F))$. 
Example of a Boundary Case

\[ F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \text{forb}(m, F) = 2m \]
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\[ [F|\alpha] = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad \text{forb}(m, [F|\alpha]) \text{ is } \Omega(m^2) \]

Construction: \( I_{m/2}^c \times I_{m/2}^c \)
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\end{bmatrix}, \quad \text{for} b(m, F) = 2m
\]

\[
[F | \alpha] = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix}, \quad \text{for} b(m, [F | \alpha]) \text{ is } \Omega(m^2)
\]

Construction: \( I_{m/2}^c \times I_{m/2}^c \) avoiding \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
Example of a Boundary Case

\[
F = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}, \quad \text{forb}(m, F) = 2m
\]

\[
[F|\alpha] = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix}, \quad \text{forb}(m, [F|\alpha]) \text{ is } \Omega(m^2)
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Construction: \( I_{m/2} \times I_{m/2} \).
Example of a Boundary Case

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Example of a Boundary Case

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Construction: \( I_{m/2} \times I_{m/2} \).
Example of a Boundary Case

\[ F = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}, \quad \text{forb}(m, F) = 2m \]

Thus \( F \) is a boundary case.
Using a result of A and Fleming 10, there are three simple column-maximal 4-rowed $F$ for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}$$

How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.
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1 & 0 & 1 & 0 & 1 & 0 \\
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0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}$$

How can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed $t$, let

$$F_8(t) = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
0 & 0
\end{bmatrix}$$
Theorem (A, Raggi, Sali 09) Let $t$ be given. Then $\text{forb}(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column $\alpha$ not already present $t$ times in $F_8(t)$, then $\text{forb}(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.
The Conjecture predicts nine 5-rowed simple matrices $F$ to be boundary cases, namely $\text{forb}(m, F)$ is predicted to be $\Theta(m^2)$ and for any column $\alpha$ we have $\text{forb}(m, [F|\alpha])$ being $\Omega(m^3)$. We have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

**Theorem** (A, Raggi, Sali) $\text{forb}(m, F_7)$ is $\Theta(m^2)$. Moreover $F_7$ is a boundary case, namely for any column $\alpha$, then $\text{forb}(m, [F_7|\alpha])$ is $\Omega(m^3)$. 
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\[
F_7 = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

**Theorem** (A, Raggi, Sali) $\text{forb}(m, F_7)$ is $\Theta(m^2)$. Moreover $F_7$ is a boundary case, namely for any column $\alpha$, then $\text{forb}(m, [F_7|\alpha])$ is $\Omega(m^3)$.

The proof is currently a rather complicated induction.
Theorem (A, Raggi, Sali) \( \text{forb}(m, G_{6 \times 3}) \) is \( \Theta(m^2) \). Moreover \( G_{6 \times 3} \) is a boundary case, namely for any column \( \alpha \), then 
\[ \text{forb}(m, [G_{6 \times 3} | \alpha]) \] is \( \Omega(m^3) \). In fact if \( F \) is not a configuration in \( G_{6 \times 3} \), then \( \text{forb}(m, F) \) is \( \Omega(m^3) \).

Proof: We use induction and the bound for \( F_7 \).
Theorem (Balogh and Bollabás 05) Given $k$, there exists a constant $c_k$ so that $\text{forb}(m, \{I_k, I_k^c, T_k\}) = c_k$. 

Theorem (A. and Meehan 11) Let $p, k$ be given with $p \geq 3k$. Let $F = \left[0_k \mid I_k \right] \times \left[0_k \mid T_k \right] \times \left[I_k^c \mid 1_k \right] \times K_{p-3k}$. Then $\text{forb}(m, F)$ is $\Theta(m^{p-k})$. Moreover $F$ is column maximal (a weak form of a boundary case), namely for any column $\alpha$ not in $F$ we have that $\text{forb}(m, [F \mid \alpha])$ is $\Omega(m^{p-k+1})$. 

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**Theorem** (Balogh and Bollabás 05) Given \( k \), there exists a constant \( c_k \) so that \( \text{forb}(m, \{I_k, I^c_k, T_k\}) = c_k \).

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Let \( F = [0_k|I_k] \times [0_k|T_k] \times [I^c_k|1_k] \times K_{p-3k} \).

Then \( \text{forb}(m, F) \) is \( \Theta(m^{p-k}) \).
**Theorem** (Balogh and Bollabás 05) Given $k$, there exists a constant $c_k$ so that $\text{forb}(m, \{I_k, I^c_k, T_k\}) = c_k$.

**Theorem** (A. and Meehan 11) Let $p, k$ be given with $p \geq 3k$. Let $F = [\mathbf{0}_k | I_k] \times [\mathbf{0}_k | T_k] \times [I^c_k | \mathbf{1}_k] \times K_{p-3k}$. Then $\text{forb}(m, F)$ is $\Theta(m^{p-k})$.

Moreover $F$ is **column maximal** (a weak form of a boundary case), namely for any column $\alpha$ not in $F$ we have that $\text{forb}(m, [F|\alpha])$ is $\Omega(m^{p-k+1})$. 
Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{cc}
\text{row } r & \begin{bmatrix}
0 & \cdots & 0 & 1 & \cdots & 1 \\
B_r & C_r & C_r & D_r
\end{bmatrix}.
\end{array}$$
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Now $[B_r C_r D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$. 

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\( C_r \) has no \( F \) in

\[
\mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \right\}
\]
Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \left[ \begin{array}{cccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{array} \right].$$

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$$\|A\| = \text{forb}(m, F_7) = \|B_rC_rD_r\| + \|C_r\| \leq \text{forb}(m - 1, F_7) + \|C_r\|.$$
Let $A$ be an $m \times forb(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{bmatrix}. $$

Now $[B_r C_r D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $F$ where $F$ is derived from $F_7$. Then

$$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m - 1, F_7) + \|C_r\|.$$

To show $forb(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of $r$. 
Repeated Induction

Let $C_r$ be an $(m - 1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s_i$ and reorder rows and columns to obtain

$$C_r = \text{row } s_i \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & G_i & G_i & H_i \end{bmatrix}.$$  

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of $s_i$. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in a contradiction.
Let $C_r$ be an $(m - 1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s_i$ and reorder rows and columns to obtain

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**Idea:** Select a minimal set of rows $L_i$ so that $G_i|_{L_i}$ is simple.
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Then we discover:

$$C_r = \text{row } s_i \left[ \begin{array}{ccccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & G_i & G_i & H_i \\ \end{array} \right]_{L_i}\{ \text{columns} \subseteq [0|I] \} L_i$$
Idea: Select a minimal set of rows $L_i$ so that $G_i|_{L_i}$ is simple.

We first discover $G_i|_{L_i} = [0|\text{I}]$ or $[1|\text{I}^c]$ or $[0|\text{T}]$.

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0 & \cdots & 0 & 1 & \cdots & 1 \\
E_i & G_i & G_i & H_i \\
\text{columns } \subseteq [1|\text{I}^c]
\end{bmatrix} L_i \{ L_i \}$$
Idea: Select a minimal set of rows $L_i$ so that $G_i|_{L_i}$ is simple.

We first discover $G_i|_{L_i} = [0|I]$ or $[1|I^c]$ or $[0|T]$.
Then we discover:

$$C_r = \text{row } s_i \left[ \begin{array}{ccccccc}
0 & \cdots & 0 & 1 & \cdots & 1 \\
E_i & G_i & G_i & H_i \\
\text{columns } \subseteq [0|T]
\end{array} \right]_{L_i \{ \_ \}}.$$
We may choose $s_1$ and form $L_1$.  
Then choose $s_2 \in L_1$ and form $L_2$.  
Then choose $s_3 \in L_2$ and form $L_3$.  
etc.
We can show the sets $L_1 \setminus s_2, L_2 \setminus s_3, L_3 \setminus s_4, \ldots$ are disjoint.  
Assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in $|L_i \setminus s_{i+1}| \geq 3$ which yields a contradiction.
THANKS to the organizers, particularly Sali Attila!