

# Forbidden Configurations: A Survey

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# Introduction

Forbidden configurations are first described as a problem area in a 1985 paper. The subsequent work has involved a number of coauthors: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Peter Keevash and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of VC-dimension uses a forbidden configuration.

Survey at [www.math.ubc.ca/~anstee](http://www.math.ubc.ca/~anstee)

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**Definition** We say  $\mathcal{F} \subseteq 2^{[m]}$  is **intersecting** if for every pair  $A, B \in \mathcal{F}$ , we have  $|A \cap B| \geq 1$ .

**Theorem** If  $\mathcal{F} \subseteq 2^{[m]}$  and  $\mathcal{F}$  is intersecting, then

$$|\mathcal{F}| \leq 2^{m-1}.$$

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i.e. if  $A$  is  $m$ -rowed and simple then  $A$  is the incidence matrix of some  $\mathcal{F} \subseteq 2^{[m]}$ .

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{F} = \{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$$

**Definition** Given a matrix  $F$ , we say that  $A$  has  $F$  as a *configuration* if there is a submatrix of  $A$  which is a row and column permutation of  $F$ .

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = A$$

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We consider the property of forbidding a configuration  $F$  in  $A$  for which we say  $F$  is a *forbidden configuration* in  $A$ .

**Definition** Let  $\text{forb}(m, F)$  be the largest function of  $m$  and  $F$  so that there exist a  $m \times \text{forb}(m, F)$  simple matrix with *no* configuration  $F$ . Thus if  $A$  is any  $m \times (\text{forb}(m, F) + 1)$  simple matrix then  $A$  contains  $F$  as a configuration.

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For example,  $\text{forb}(m, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 2$ ,  $\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$ .

**Definition** Let  $K_k$  denote the  $k \times 2^k$  simple matrix of all possible columns on  $k$  rows (i.e. incidence matrix of  $2^{[k]}$ ).

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

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## Two interesting examples

Let

$$F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\text{forb}(m, F_1) = 2m, \quad \text{forb}(m, F_2) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

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**Problem** *What drives the asymptotics of  $\text{forb}(m, F)$ ? What structures in  $F$  are important?*

The building blocks of our constructions are  $I$ ,  $I^c$  and  $T$ :

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

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**Definition** Given an  $m_1 \times n_1$  matrix  $A$  and a  $m_2 \times n_2$  matrix  $B$  we define the product  $A \times B$  as the  $(m_1 + m_2) \times (n_1 n_2)$  matrix consisting of all  $n_1 n_2$  possible columns formed from placing a column of  $A$  on top of a column of  $B$ . If  $A, B$  are simple, then  $A \times B$  is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given  $p$  simple matrices  $A_1, A_2, \dots, A_p$ , each of size  $m/p \times m/p$ , the  $p$ -fold product  $A_1 \times A_2 \times \dots \times A_p$  is a simple matrix of size  $m \times (m^p/p^p)$  i.e.  $\Theta(m^p)$  columns.

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# The Conjecture

**Definition** Let  $x(F)$  denote the largest  $p$  such that there is a  $p$ -fold product which does not contain  $F$  as a configuration where the  $p$ -fold product is  $A_1 \times A_2 \times \cdots \times A_p$  where each  $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$ .

Thus  $x(F) + 1$  is the smallest value of  $p$  such that  $F$  is a configuration in every  $p$ -fold product  $A_1 \times A_2 \times \cdots \times A_p$  where each  $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$ .

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**Conjecture** (A, Sali 05)  $\text{forb}(m, F)$  is  $\Theta(m^{x(F)})$ .

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The conjecture has been verified for  $k \times I \ F$  where  $k = 2$  (A, Griggs, Sali 97) and  $k = 3$  (A, Sali 05) and  $I = 2$  (A, Keevash 06) and other cases.

**Theorem** (Balogh, Bollabás 05) *Given  $k$ , there is a constant  $c_k$  with*

$$\text{forb}(m, \{I_k, I_k^c, T_k\}) = c_k.$$

Can we get good bounds on this constant?

# Refinements of the Sauer Bound

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)  $\text{forb}(m, K_k)$  is  $\Theta(m^{k-1})$

Let  $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

**Theorem** (A, Fleming) *Let  $F$  be a  $k \times l$  simple matrix such that there is a pair of rows with no configuration  $E_1$  and there is a pair of rows with no configuration  $E_2$  and there is a pair of rows with no configuration  $E_3$ . Then  $\text{forb}(m, F)$  is  $O(m^{k-2})$ .*

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Note that  $F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  has no  $E_1$  on rows 1,3, no  $E_2$  on rows 1,2 and no  $E_3$  on rows 2,3. Thus  $\text{forb}(m, F_1)$  is  $O(m)$ .

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**Theorem** (A, Fleming) *Let  $E$  be given with  $E \in \{E_1, E_2, E_3\}$ . Let  $F$  be a  $k \times l$  simple matrix with the property that every pair of rows contains the configuration  $E$ . Then  $\text{forb}(m, F) = \Theta(m^{k-1})$ .*

$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$  has  $E_3$  on rows 1,2.

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Note that  $F_2$  has  $E_3$  on every pair of rows hence  $\text{forb}(m, F_2)$  is  $\Theta(m^2)$  (A, Griggs, Sali 97).

In particular, this means  $F_2 \notin T \times T$  which is the construction to achieve the bound.

**Definition** Let  $t \cdot A$  be the matrix consisting of  $t$  copies of  $A$  placed side by side.

**Theorem** (Gronau 80)

$$\text{forb}(m, 2 \cdot K_k) = \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0} = \Theta(m^k)$$

**Theorem** (A, Füredi 86)

$$\text{forb}(m, t \cdot K_k) = \frac{t-2}{k+1} \binom{m}{k} (1+o(1)) + \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}$$

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**Theorem** (A, Griggs, Sali 97, A, Sali 05,  
A, Fleming, Füredi, Sali 05)  
*forb(m, F\_B(t)) is  $\Theta(m^{k-1})$ .*

The difficult problem here was the bound although induction works.

Let  $A$  be given and let  $S \in \binom{[m]}{k}$ . We define  $A|_S$  to be the submatrix of  $A$  given by the  $k$  rows indexed by  $S$ . For a given  $k \times 1$   $(0,1)$ -column  $\alpha$ , use the notation

$\#\alpha$  in  $A|_S$

to be the number of columns of  $A|_S$  equal to  $\alpha$  (not permuting rows).

**Theorem** (A, Fleming, Füredi, Sali 05) *Let  $A$  be an  $m$ -rowed simple matrix and let  $\mathcal{S} \subseteq \binom{[m]}{k}$  where for each  $S \in \mathcal{S}$  there are two  $k \times 1$   $(0,1)$ -columns  $\alpha_S, \beta_S$  with  $\#\alpha_S$  in  $A|_S \leq t$  and  $\#\beta_S$  in  $A|_S \leq t$ .*

*Then the number of columns  $\gamma$  in  $A$  such that there exists an  $S \in \mathcal{S}$  with  $\gamma|_S = \alpha_S$  or  $\gamma|_S = \beta_S$ , is at most  $2t \left( \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \right)$ .*

$$\text{Let } F_3(t) = \left[ \begin{array}{c|ccccc} 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array} \right] t \cdot \left[ \begin{array}{ccccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array} \right].$$

**Theorem** (A, Sali 05) *forb*( $m, F_3(t)$ ) is  $\Theta(m^2)$ .

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$$\text{Let } F_k(t) = \left[ \begin{array}{c|c} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & t \cdot G_k \end{array} \right],$$

where  $G_k$  is the  $k \times (2^k - 2^{k-2} - 1)$  simple matrix of all columns with at least one 1 and not having 1's in both rows 1 and 2 at the same time.

**Conjecture** (A, Sali 05)  $\text{forb}(m, F_k(t))$  is  $\Theta(m^{k-1})$ .

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The standard induction does not work here.

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**Theorem** (A, Sali 05)  $\text{forb}(m, F_3(t))$  is  $\Theta(m^2)$ .

$$\text{Let } F_k(t) = \left[ \begin{array}{c|c} 0 & \\ \vdots & t \cdot G_k \\ 0 & \end{array} \right],$$

where  $G_k$  is the  $k \times (2^k - 2^{k-2} - 1)$  simple matrix of all columns with at least one 1 and not having 1's in both rows 1 and 2 at the same time.

**Conjecture** (A, Sali 05)  $\text{forb}(m, F_k(t))$  is  $\Theta(m^{k-1})$ .

The standard induction does not work here.

**Theorem** (A, Sali 05)  $\text{forb}(m, t \cdot I_k)$  is  $\Theta(m^{k-1})$ .

# Intersecting Families and $k \times 2 F$

**Definition** We say  $\mathcal{F} \subseteq 2^{[m]}$  is  **$t$ -intersecting** if for every pair  $A, B \in \mathcal{F}$ , we have  $|A \cap B| \geq t$ .

**Theorem** (Katona 64) *Let  $t, m$  be given. Let  $\mathcal{F} \subseteq 2^{[m]}$  be a maximum sized  $t$ -intersecting family. Then  $\mathcal{F} = \mathcal{K}_{m,t}$  where*

$$\mathcal{K}_{m,t} = \begin{cases} \{A \subseteq [m] : |A| \geq (m+t)/2\} & \text{if } m+t \text{ is even} \\ \{A \subseteq [m] : |A \setminus \{1\}| \geq (m+t-1)/2\} & \text{if } m+t \text{ is odd} \end{cases}$$

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**Theorem** (Ahlswede and Khachatrian 97)

*Complete Intersection Theorem.*

Let  $m, k, r$  be given. A maximum sized  $(k - r)$ -intersecting  $k$ -uniform family  $\mathcal{F} \subseteq \binom{[m]}{k}$  is isomorphic to  $\mathcal{I}_{r_1, r_2}$  for some choice  $r_1 + r_2 = r$  and for some choice  $G \subseteq [m]$  where  $|G| = k - r_1 + r_2$  where

$$\mathcal{I}_{r_1, r_2} = \{A \subseteq \binom{[m]}{k} : |A \cap G| \geq k - r_1\}$$

This generalizes the Erdős-Ko-Rado Theorem (61).

**Theorem** (A-Keevash 06) Stability Lemma.

Let  $r$  be given. Let  $\mathcal{F} \subseteq \binom{[m]}{k}$ .

Assume that  $\mathcal{F}$  is  $(k-r)$ -intersecting and

$$|\mathcal{F}| \geq (6r)^{5r+7} m^{r-1}.$$

Then  $\mathcal{F} \subseteq \mathcal{I}_{r_1, r_2}$  for some choice  $r_1 + r_2 = r$  and for some choice  $G \subseteq [m]$  where  $|G| = k - r_1 + r_2$ .

This result is for large intersections; we use it with a fixed  $r$  where  $k$  can grow with  $m$ .

**Definition** Let  $F_{a,b,c,d}$  denote the  $(a + b + c + d) \times 2$  matrix of  $a$  rows  $[11]$ ,  $b$  rows of  $[10]$ ,  $c$  rows of  $[01]$ , and  $d$  rows of  $[00]$ . We assume  $a \geq d$  and  $b \geq c$ .

**Theorem** (A-Keevash 06) *if  $b > c$  or  $a, b \geq 1$ , then*

$$\text{forb}(m, F_{a,b,c,d}) = \Theta(m^{a+b-1}).$$

*Also  $\text{forb}(m, F_{0,b,b,0}) = \Theta(m^b)$  and  $\text{forb}(m, F_{a,0,0,d}) = \Theta(m^a)$ .*

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**Proof:** The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

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**Proof:** The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

The theorem is further evidence for the conjecture.

e.g. Let  $A$  be a simple matrix with no  $F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Let  $A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ B_1 & B_2 & B_2 & B_3 \end{bmatrix}$  (the **standard induction**),

where  $B_2$  is chosen to be all columns which are repeated after deleting row 1 of  $A$ .

Then  $[B_1 B_2 B_3]$  is simple and has no  $F_{0,3,2,0}$  and so by induction has at most  $c(m-1)^2$  columns.  $B_2$  is also simple and we verify that  $B_2$  has at most  $cm$  columns and so by induction  $\text{forb}(m, F_{0,3,2,0}) \leq c(m-1)^2 + cm \leq cm^2$ .

$$\text{Given } A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ B_1 & B_2 & B_2 & B_3 \end{bmatrix} \text{ with no } F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\text{then } B_2 \text{ has no } F_{0,2,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } F_{0,3,1,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Our proof then uses the fact that if we only consider the columns of column sum  $k$  in  $B_2$  as a set system, then using the fact that  $F_{0,2,2,0}$  is forbidden we deduce that the  $k$ -uniform set system is  $(k-1)$ -intersecting. We then use our stability result to either determine the columns have a certain structure or that the bound is true because there are so few columns.

# Some Exact Bounds

A, Griggs, Sali 97, A, Ferguson, Sali 01, A, Kamoosi 07  
A, Barekat, Sali 08, A, Barekat 08

Configuration $F$	Exact Bound $\text{forb}(m, F)$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	2
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$m + 2$
$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$2m + 2$
$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{5m}{2} \rfloor + 2$
$q \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lfloor \frac{(q+1)m}{2} \rfloor + 2$ , for $m$ large

Configuration $F$	Exact Bound $\text{forb}(m, F)$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{3m}{2} \rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{7m}{3} \rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{11m}{4} \rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{15m}{4} \rfloor + 1$

Configuration $F$	Exact Bound for $b(m, F)$
$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{8m}{3} \rfloor$
$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{10m}{3} - \frac{4}{3} \rfloor$
$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$4m$
$\begin{bmatrix} \overbrace{1 \dots 1}^p & \overbrace{0 \dots 0}^p \\ 0 \dots 0 & 1 \dots 1 \end{bmatrix}$	$pm - p + 2$

$$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Theorem** (A, Dunwoody)  $\text{forb}(m, F_2) = \lfloor \frac{m^2}{4} \rfloor + m + 1$

**Proof:** The proof technique is that of shifting, popularized by Frankl. A paper of Alon 83 using shifting refers to the possibility of such a result.

Configuration F	$\text{forb}(m, F)$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + m - 2$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\leq \frac{3}{2} \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ $\geq \frac{29}{21} \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ <p>no exact coefficient of <math>\binom{m}{2}</math> possible</p>

Configuration F	$\text{forb}(m, F)$
$\begin{array}{c} \overbrace{\phantom{1\ 1\ \dots\ 1}}^p \\ \left[ \begin{array}{c} 1\ 1\ \dots\ 1 \\ 1\ 1\ \dots\ 1 \\ 0\ 0\ \dots\ 0 \\ 0\ 0\ \dots\ 0 \end{array} \right] \end{array}$	$\frac{p+3}{3} \binom{m}{2} + 2 \binom{m}{1} + 2 \binom{m}{0}$ <p>for <math>m</math> large, <math>m \equiv 1, 3 \pmod{6}</math></p>

THANKS FOR THE CHANCE TO VISIT COLLEGE OF  
CHARLESTON!