

Forbidden Configurations: Progress on a Conjecture

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Definition We say that a matrix A is *simple* if it is a $(0,1)$ -matrix with no repeated columns.

i.e. if A is m -rowed then A is the incidence matrix of some family \mathcal{A} of subsets of $[m] = \{1, 2, \dots, m\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}$$

Definition We define $\|A\|$ to be the number of columns in A .

$$\|A\| = 6$$

Definition Given a matrix F , we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F .

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & \boxed{0} & \boxed{1} & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1} & \boxed{1} & \boxed{0} & 0 & \boxed{0} \end{bmatrix}$$

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We consider the property of forbidding a configuration F in A .

Definition Let

$forb(m, F) = \max\{\|A\| : A \text{ } m\text{-rowed simple, no configuration } F\}$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

$$\text{forb}(m, K_k) = \max\{\|A\| : A \text{ has VC - dimension } k - 1\}$$

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Corollary Let F be a $k \times \ell$ simple matrix. Then $\text{forb}(m, F) = O(m^{k-1})$.

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Theorem (Füredi 83). Let F be a $k \times \ell$ matrix. Then $\text{forb}(m, F) = O(m^k)$.

A Product Construction

The building blocks of our product constructions are I , I^c and T , e.g:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition Given two matrices A, B , we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of A on top of a column of B in all possible ways. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Given p simple matrices A_1, A_2, \dots, A_p , each of size $m/p \times m/p$, the p -fold product $A_1 \times A_2 \times \dots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.

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$$[01] \times [01] = K_2$$

$$\overbrace{[01] \times [01] \times \cdots \times [01]}^k = K_k$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2, m/2}$

The Conjecture

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Definition Let F be given. Let $x(F)$ denote the largest p such that there is a p -fold product which does not contain F as a configuration where the p -fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$.

Conjecture (A, Sali 05) $\text{forb}(m, F)$ is $\Theta(m^{x(F)})$.

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for k -rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 10) plus other cases.

Forbidden Families are Difficult

Let G be a given graph. We define $\text{ex}(m, G)$ to be the maximum number of edges in a graph on m vertices which has no subgraph isomorphic to G . Let F denote the vertex-edge incidence matrix of graph G . Then

$$\text{forb}\left(m, \left\{ F, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right) = \text{ex}(m, G) + m + 1.$$

Theorem (Balogh and Bollobás 05) *Given k , there exists a constant c_k so that $\text{forb}(m, \{I_k, I_k^c, T_k\}) = c_k$.*

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Theorem (A. and Meehan 11) *Let p, k be given with $p \geq 3k$. Let $F = [0_k | I_k] \times [0_k | T_k] \times [I_k^c | 1_k] \times K_{p-3k}$. Then $\text{forb}(m, F)$ is $\Theta(m^{p-k})$.*

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Then $\text{forb}(m, F)$ is $\Theta(m^{p-k})$.

*Moreover F is a **boundary case**, namely for any column α not in F we have that $\text{forb}(m, [F | \alpha])$ is $\Omega(m^{p-k+1})$.*

Using a result of A. and Fleming 10, there are three simple **column-maximal** 4-rowed F for which $\text{forb}(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

$$F_8(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Theorem (A, Raggi, Sali 09) *Let t be given. Then $\text{forb}(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a **boundary case**, namely for any column α not already present t times in $F_8(t)$, then $\text{forb}(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.*

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

5×6 Simple Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices F which are **boundary cases**, namely $\text{forb}(m, F)$ is predicted to be $\Theta(m^2)$ and for any column α we have $\text{forb}(m, [F|\alpha])$ being $\Omega(m^3)$. Such F happen all to be 5×6 simple matrices and we have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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All 6-rowed Configurations with Quadratic Bounds

$$G_{6 \times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem (A,Raggi,Sali) *Let F be any 6-rowed configuration. Then $\text{forb}(m, F)$ is $\Theta(m^2)$ if F is a configuration in $G_{6 \times 3}$ and $\text{forb}(m, F)$ is $\Omega(m^3)$ if F is not a configuration in $G_{6 \times 3}$.*

Proof: We use induction and the bound for F_7 .

Induction

Let A be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration F_7 . We can select a row r and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{bmatrix}.$$

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Now $[B_r C_r D_r]$ is an $(m - 1)$ -rowed simple matrix with no configuration F_7 . Also C_r is an $(m - 1)$ -rowed simple matrix with no configurations in \mathcal{F} where \mathcal{F} is derived from F_7 .

C_r has no F in

$$\mathcal{F} = \left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \right\}$$

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Then

$$\|A\| = \text{forb}(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq \text{forb}(m-1, F_7) + \|C_r\|.$$

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$$\|A\| = \text{forb}(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq \text{forb}(m-1, F_7) + \|C_r\|.$$

To show $\text{forb}(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of r .

Repeated Induction

Let C_r be an $(m - 1)$ -rowed simple matrix with no configuration in \mathcal{F} . We can select a row s_j and reorder rows and columns to obtain

$$C_r = \text{row } s_j \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_j & & G_j & G_j & & H_j \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_j\|$ is bounded by a constant for some choice of s_j . Our proof shows that assuming $\|G_j\| \geq 8$ for all choices s_j results in a contradiction.

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We first discover $G_i|_{L_i} = [\mathbf{0}|I]$ or $[\mathbf{1}|I^c]$ or $[\mathbf{0}|T]$.

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Then we discover:

$$C_r = \text{row } s_j \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & & G_i & G_i & & H_i \end{array} \right]_{L_i \left\{ \begin{array}{l} \text{columns} \subseteq [\mathbf{0}|I] \end{array} \right\} L_i} .$$

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We may choose s_1 and form L_1 .

Then choose $s_2 \in L_1$ and form L_2 .

Then choose $s_3 \in L_2$ and form L_3 .

etc.

We can show the sets $L_1 \setminus s_2, L_2 \setminus s_3, L_3 \setminus s_4, \dots$ are disjoint.

Assuming $\|G_i\| \geq 8$ for all choices s_i results in $|L_i \setminus s_{i+1}| \geq 3$ which yields a contradiction.

An unusual Bound

Theorem (A,Raggi,Sali 10) $\text{forb}(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad T_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Let A be an $m \times \text{forb}(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\}$. We can select a row r and reorder rows and columns to obtain

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To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of r . Our proof shows that assuming $\|C_r\| > 36m^{1/2}$ for all choices r results in a contradiction.

THANKS to University of Victoria for hosting this conference!

We can extend K_4 and still have the same bound

$$[K_4 | \mathbf{1}_2 \mathbf{0}_2] =$$

$$\left[\begin{array}{ccccccccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

Theorem (A., Meehan) For $m \geq 5$, we have
 $\text{forb}(m, [K_4 | \mathbf{1}_2 \mathbf{0}_2]) = \text{forb}(m, K_4)$.

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Theorem (A., Meehan) For $m \geq 5$, we have
 $forb(m, [K_4 | \mathbf{1}_2 \mathbf{0}_2]) = forb(m, K_4)$.

We expect in fact that we could add many copies of the column $\mathbf{1}_2 \mathbf{0}_2$ and obtain the same bound, albeit for larger values of m .