Forbidden Configurations and Indicator Polynomials

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BIRS, Invariants of Incidence Matrices, April 3, 2009
The use of indicator polynomials was explored in a joint paper with Fleming, Füredi and Sali. This talk focuses on joint work with Balin Fleming that led to a breakthrough for Forbidden Configurations. Füredi and Sali continue to explore applications to critical hypergraphs.

Forbidden Configuration Survey at [www.math.ubc.ca/~anstee](http://www.math.ubc.ca/~anstee)
**Definition** We say that a matrix $A$ is *simple* if it is a $(0,1)$-matrix with no repeated columns. We can think of $A$ as the incidence matrix of some set system $\mathcal{F}$.

$[m] = \{1, 2, \ldots, m\}$

Let $S$ be a finite set.

$2^S = \{T : T \subseteq S\}$

$(S)_k = \{T \in 2^S : |T| = k\}$
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i.e. if $A$ is an $m$-rowed simple matrix then $A$ is the incidence matrix of some $\mathcal{F} \subseteq 2^m$. 

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Some matrix notations are helpful:

- $K_k$ is the $k \times 2^k$ simple matrix $\approx 2^{[k]}$
Definition  Given a matrix $F$, we say that $A$ has $F$ as a configuration if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$
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We consider the property of forbidding a configuration $F$ in $A$ for which we say $F$ is a *forbidden configuration* in $A$.

**Definition** Let $\text{forb}(m, F)$ be the largest function of $m$ and $F$ so that there exist a $m \times \text{forb}(m, F)$ simple matrix with no configuration $F$. Thus if $A$ is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then $A$ contains $F$ as a configuration.
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For example, $\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) = m + 1$. 
**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows (i.e. incidence matrix of $2^{[k]}$).

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$
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**Corollary** Let $F$ be a $k \times 1$ simple matrix. Then

$$forb(m, F) = O(m^{k-1})$$
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$$forb(m, F) = O(m^k)$$
Order Shattered Sets

Let \( \mathcal{F} \subseteq 2^m \). We say \( S = \{i_1, i_2, i_3\} \) is order-shattered by \( \mathcal{F} \) (or the associated incidence matrix \( A \)) if there are \( 2^3 \) columns of \( A \)

\[
\begin{bmatrix}
* & * & * & * & * & * & * & * & * & * \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\delta & \delta & \epsilon & \epsilon & \kappa & \kappa & \lambda & \lambda & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\beta & \beta & \beta & \beta & \gamma & \gamma & \gamma & \gamma & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha
\end{bmatrix}
\]

\[\rightarrow \text{ row } i_1\]

\[\rightarrow \text{ row } i_2\]

\[\rightarrow \text{ row } i_3\]

Note that \( A|_S \) has \( K_3 \). The symbols \( \alpha, \beta, \ldots, \lambda \) are for vectors of appropriate length and \( * \) refers to arbitrary entries.
Using the definition of order shattered sets we define

\[ osh(\mathcal{F}) = \{ S \in 2^m \mid S \text{ is order shattered by } \mathcal{F} \} \]

The set \( osh(\mathcal{F}) \) is a downset and moreover \( |osh(\mathcal{F})| = |\mathcal{F}| \).

**Theorem** (A, Ronyai, Sali 02) The inclusion matrix \( I(osh(\mathcal{F}), \mathcal{F}) \) is nonsingular over every field.
Let $A$ be an $m$-rowed simple matrix which has no configuration $K_k$. For any $k$-set of rows $S \in \binom{[m]}{k}$, we let $A|_S$ denote the submatrix of $A$ given by the rows of $S$. Since $A$ has no $K_k$, then for every $k$-set $S$ of rows we have that $A|_S$ has an absent $k \times 1$ $(0,1)$-column.

**Remark** If $A$ has the property that for every $k$-set of rows $S \in \binom{[m]}{k}$ we have that $A|_S$ has an absent column, then $A$ has no $K_k$ and so has at most $O(m^{k-1})$ columns.
Let $B$ be a $k \times (k + 1)$ simple matrix with one column of each column sum. For a matrix $C$, let $t \cdot C$ denote the matrix $[CCC \cdots C]$ from concatenating $t$ copies of $C$. Let

$$F_B(t) = [K_k \ t \cdot [K_k \setminus B]]$$

Let $k, t$ be given. Let $A$ be any $m$-rowed simple matrix which has no configuration $F_B(t)$. Then for any $k$-set of rows $S \in \binom{[m]}{k}$, either $A|_S$ has an absent column or $A|_S$ has two columns which appear at most $t$ times each.

*We say such columns are in short supply.*
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We say such columns are in short supply.

Idea: We wish to show that we could delete $O(m^{k-1})$ columns from $A$ to obtain $A'$ where $A'$ has an absent column for each $k$-set of rows and hence $A'$ has at most $O(m^{k-1})$ columns and so $A$ has at most $O(m^{k-1})$ columns.
Assume $A$ is an $m$-rowed simple matrix with no $F_B(t)$.
Let $S \in \binom{[m]}{k}$ and let $\alpha$ be a $k \times 1$ column which is in short supply on $S$.
We say that an $m \times 1$ column $\gamma$ violates $S$ (for the chosen $\alpha$) if
\[ \gamma|S = \alpha. \]
Assume $A$ is an $m$-rowed simple matrix with no $F_B(t)$.

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$$\gamma|_S = \alpha.$$

Let $T \subseteq \binom{[m]}{k}$ be the set of $k$-sets $S$ for which there are (at least) two $k \times 1$ columns $\alpha, \beta$ in short supply (no column absent).

We could eliminate $\leq t|T|$ columns with violations on $S \in T$ from $A$ to obtain $A'$ which has an absent column on each $k$-set of rows.

Unfortunately $|T|$ can be too large. We need a better way to estimate the number of columns in $A$ that have violations on $S \in T$. 
Let $S = \{i_1, i_2, \ldots, i_k\} \in \binom{[m]}{k}$

Let $x_1, x_2, \ldots, x_m$ be variables. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^T$ be a $k \times 1$ $(0,1)$-column. Let $a$ be the number of $1$'s in $\alpha$.

$$f_{S, \alpha}(\mathbf{x}) = \prod_{j=1}^{k} (x_{i_j} - \alpha_j)$$

For a $m \times 1$ $(0,1)$-column $\gamma$

$$f_{S, \alpha}(\bar{\gamma}) = \begin{cases} (-1)^a & \text{if } \gamma|S = \alpha \\ 0 & \text{otherwise} \end{cases}$$
Let $S = \{i_1, i_2, \ldots, i_k\} \in \binom{[m]}{k}$

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$$f_{S,\alpha}(x) = \prod_{j=1}^{k} (x_{i_j} - \alpha_j)$$

For a $m \times 1$ $(0,1)$-column $\gamma$

$$f_{S,\alpha}(\gamma) = \begin{cases} (-1)^a & \text{if } \gamma|_S = \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$f_{S,\alpha}(\overline{\gamma}) = \begin{cases} \neq 0 & \text{if } \gamma|_S = \alpha \\ 0 & \text{otherwise} \end{cases}$$
Multilinear Indicator Polynomials

\[ f_{S,\alpha}(x) = \prod_{j=1}^{k} (x_{ij} - \alpha_j) \]

We check that degree of \( f_{S,\alpha}(x) \) is \( k \) with leading term

\[ \prod_{j=1}^{k} x_{ij} \]
Assume $S \in \mathcal{T}$ and there are two $k \times 1$ columns $\alpha, \beta$ in short supply (no column absent) and the two indicator polynomials $f_{S,\alpha}, f_{S,\beta}$. We set

$$f_S(x) = a_1 f_{S,\alpha}(x) + a_2 f_{S,\beta}(x)$$

$$a_1 = +1, \quad a_2 = -1$$

We have that for a $m \times 1$ (0,1)-column $\gamma$

$$f_S(\bar{\gamma}) \begin{cases} 
\neq 0 & \text{if } \gamma|_S = \alpha \text{ or } \gamma|_S = \beta \\
= 0 & \text{otherwise}
\end{cases}$$

and degree of $f_S(x)$ is (at most) $k - 1$ since the leading terms of degree $k$ of $f_{S,\alpha}(x)$ and $f_{S,\beta}(x)$ will cancel.
Let $\mathcal{I} \subseteq \binom{[m]}{k}$ be the set of all $k$-tuples $S$ for which two columns, say $\alpha, \beta$ are in short supply.

An independent set $I = (S_i)$ is an ordered list $S_1, S_2, \ldots \in \mathcal{I}$ and $k \times 1$ $(0,1)$-columns $\gamma_1, \gamma_2, \ldots$ of $A$ so that

$\gamma_i$ violates $S_i$ for two chosen columns but violates no $S_j$ with $j < i$

An independent set can be found by a greedy approach.
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An independent set can be found by a greedy approach.

**Theorem** If $\mathcal{I} = (S_i)$ is an independent set, then the indicator polynomials $f_S$ are linearly independent.

**Proof:** Form the matrix of order $|\mathcal{I}|$ with $ij$ entry equal to

$$f_{S_j}(\vec{\gamma}_i)$$

The matrix is upper triangular with nonzeros on the diagonal.
Assume we have an independent set $\mathcal{I} = (S_i)$. 
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**Theorem** If \( \mathcal{I} = (S_i) \) is an independent set, and the indicator polynomials \( f_S \) are degree at most \( d \) then

\[
|\mathcal{I}| \leq \binom{m}{d} + \binom{m}{d-1} + \cdots + \binom{m}{0} = \Theta(m^d).
\]
Assume we have an independent set $\mathcal{I} = (S_i)$.

**Theorem** If $\mathcal{I} = (S_i)$ is an independent set, and the indicator polynomials $f_S$ are degree at most $d$ then

$$|\mathcal{I}| \leq \binom{m}{d} + \binom{m}{d-1} + \cdots + \binom{m}{0} = \Theta(m^d).$$

In our case the indicator polynomials have degree $k - 1$ and so $|\mathcal{I}|$ is $O(m^{k-1})$. 
Theorem \( \text{forb}(m, F_B(t)) \) is \( \Theta(m^{k-1}) \).

Proof: Assume \( A \) is a matrix with no \( F_B(t) \). For a given set \( S \in \mathcal{T} \subseteq \binom{[m]}{k} \) there are at most \( 2t \) columns with violations of the two chosen columns in short supply on \( S \). Let \( \mathcal{I} = (S_i) \) be a maximal independent set with indicator polynomials \( f_{S_i} \). Thus we may remove \( 2t|\mathcal{I}| \) or \( O(m^{k-1}) \) columns and remove all violations on the two chosen \( k \times 1 \) columns for each \( S \in \mathcal{T} \) and so on each \( S \in \mathcal{T} \) there will be an absent column. The resulting matrix has at most \( O(m^{k-1}) \) columns and so \( A \) has at most \( O(m^{k-1}) \) columns.
There is one more $k$-rowed configuration $F$, for each $k$, with \( \text{forb}(m, F) = \Theta(m^{k-1}) \). Let $k = 4$ and let $D$ be the $k \times (11)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take $F_D(t) = [0_k (t + 1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (t + 1) \cdot \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$
There is one more \( k \)-rowed configuration \( F \), for each \( k \), with \( \text{forb}(m, F) = \Theta(m^{k-1}) \). Let \( k = 4 \) and let \( D \) be the \( k \times (11) \) simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take 

\[
F_D(t) = [0_k \cdot (t + 1)] \cdot D
\]

which for \( k = 4 \) becomes

\[
F_D(t) = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & (t + 1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

If a matrix \( A \) has no \( F_D(t) \) then each 4-set of rows \( \{i_1, i_2, i_3, i_4\} \) in some ordering has one of the following occur:

- no
- \( i_1 \) 0
- \( i_2 \) 0
- \( i_3 \) 0
- \( i_4 \) 0
We have one more $k$-rowed configuration $F$, for each $k$, with $\text{forb}(m, F)$ being $O(m^{k-1})$. Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take $F_D(t) = [0_k(t + 1) \cdot D]$ which for $k = 4$ becomes

\[ F_D(t) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix} \]

If a matrix $A$ has no $F$ then each 4-set of rows $\{i_1, i_2, i_3, i_4\}$ in some ordering has one of the following occur:

- no
- $i_1, 0$
- $i_2, 0$ or at least two columns are in short supply
- $i_3, 0$
- $i_4, 0$
We have one more $k$-rowed configuration $F$, for each $k$, with $\text{forb}(m, F)$ being $O(m^{k-1})$. Let $k = 4$ and let $D$ be the $k \times 11$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take $F_D(t) = [0_4 (t + 1) \cdot D]$ which becomes

$$F_D(t) = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}$$

If a matrix $A$ has no $F$ then each 4-set of rows $\{i_1, i_2, i_3, i_4\}$ in some ordering has one of the following occur:

- no
- $i_1 = 0$
- $i_2 = 0$ or at least two columns are in short supply or $\leq t$
- $i_3 = 0$
- $i_4 = 0$
The case of one column in short supply makes the proof much more difficult. We can find ways to eliminate $O(m^3)$ columns and make a column absent on many 4-sets of rows. But some are left.
A typical situation if we avoid $F_D(t)$ could be:

<table>
<thead>
<tr>
<th></th>
<th>≤ t</th>
<th>≤ t</th>
<th>≤ t</th>
<th>≤ t</th>
<th>≤ t</th>
<th>≤ t</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$i_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$i_3$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$i_4$</td>
<td>1</td>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$i_5$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

We only need a few of these columns to get our reduction in degree. Note that

<table>
<thead>
<tr>
<th></th>
<th>≤ t</th>
<th></th>
<th>≤ t</th>
<th>≤ t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_1$</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$i_2$</td>
<td></td>
<td>→</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$i_3$</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$i_4$</td>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$i_5$</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
\(|S| = 5\)

\[
f_{S,\alpha}(x) = \prod_{j=1}^{5}(x_{ij} - \alpha_j)
\]

For a 5 \times 1 vector \(\alpha\), we check that degree of \(f_{S,\alpha}(x)\) is 5 with leading term

\[
\prod_{j=1}^{5} x_{ij}
\]

Consider \(\sum y_i f_{S,\alpha(i)}(x)\) for some 5 \times 1 columns \(\alpha(1), \alpha(2), \ldots\). We can cancel the terms of degree 5 if \(1^T y = 0\).
\[ |S| = 5 \]

\[ f_{S,\alpha}(x) = \prod_{j=1}^{5} (x_j - \alpha_j) \]

The terms of degree 4 in \( f_{S,\alpha}(x) \) are

\[ \alpha_r \prod_{j \in S \backslash r} x_j = \begin{cases} \prod_{j \in S \backslash r} x_j & \text{if } \alpha_r = 1 \\ 0 & \text{otherwise} \end{cases} \]

Consider \( \sum y_i f_{S,\alpha(i)}(x) \) for some \( 5 \times 1 \) columns \( \alpha(1), \alpha(2), \ldots \). Let \( M \) denote the matrix whose columns are the vectors \( \alpha(1), \alpha(2), \ldots \). We can cancel the terms of degree 4 if \( My = 0 \).
We are trying to find solutions to $M\mathbf{y} = 0$ with $\mathbf{1}^T\mathbf{y} = 0$. We are able to get two easy solutions to $M\mathbf{y} = 0$:

\[
\begin{array}{cccccccc}
\mathbf{y} & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
\mathbf{i}_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\mathbf{i}_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{i}_3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\mathbf{i}_4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\mathbf{i}_5 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

If we add the two solutions together, we obtain a solution $\mathbf{y}$ whose sum of coefficients is 0 i.e. $\mathbf{1}^T\mathbf{y} = 0$.

\[
\begin{array}{cccc}
\mathbf{y} & +1 & +1 & -1 \\
\mathbf{i}_1 & 0 & 0 & 0 \\
\mathbf{i}_2 & 0 & 0 & 0 \\
\mathbf{i}_3 & 0 & 0 & 0 \\
\mathbf{i}_4 & 1 & 0 & 1 \\
\mathbf{i}_5 & 0 & 1 & 1 \\
\end{array}
\]
Adding the two previous solutions together we obtain a solution to $M\mathbf{y} = 0$ with $\mathbf{1}^T\mathbf{y} = 0$:

$$
\begin{align*}
\mathbf{y} & \quad -1 & +1 & -1 & +1 & -1 & +1 \\
 i_1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 i_2 & 1 & 1 & 0 & 0 & 0 & 0 \\
 i_3 & 0 & 1 & 1 & 0 & 0 & 0 \\
 i_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
 i_5 & 1 & 1 & 1 & 0 & 0 & 1
\end{align*}
$$

We obtain an indicator polynomial $f(\mathbf{x})$ of degree 3 with $f(\overline{\gamma}) \neq 0$ if and only if $\gamma$ violates one of the 6 listed columns on 5 rows. Note that $f(\mathbf{x})$ is an indicator polynomial for the 6 columns in short supply on the 5 rows but is not an indicator polynomial for all columns in short supply (or absent) on the 5 rows.
**Theorem** \( \text{forb}(m, F_D(t)) \) is \( O(m^3) \).

**Proof:** Assume \( A \) has no \( F_D(t) \) and that we have deleted \( O(m^3) \) columns. Consider the cases above for \( S \in \binom{[m]}{5} \) and for which we have an indicator polynomial of degree at most 3 counting some violations (6 in example above).

Then we can create a maximal independent set \( \mathcal{I} = (S_i) \) as before and given that the indicator polynomials are of degree at most 3, we can eliminate \( O(m^3) \) columns. Further eliminations of \( O(m^3) \) columns are required before there is guaranteed to be an absent column on each 4-set of \( \binom{[m]}{4} \) at which point we conclude that \( O(m^3) \) columns remain and so \( A \) has at most \( O(m^3) \) columns.
**Theorem** Let $k, t$ be given positive integers with $k \geq 2$, $t \geq 1$. Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take $F_D(t) = [0_k (t + 1) \cdot D]$ Then

$$forb(m, F_D(t)) \text{ is } \Theta(m^{k-1})$$
Theorem Let $k, t$ be given positive integers with $k \geq 2, t \geq 1$. Let $D$ be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1’s in rows 1 and 2. We take $F_D(t) = [0_k (t + 1) \cdot D]$. Then

$$\text{forb}(m, F_D(t)) = \Theta(m^{k-1})$$

Theorem Let $k$ be given and assume $F$ is a $k$-rowed configuration which is not a configuration in $F_B(t)$ (for any choice of $B$ as a $k \times (k + 1)$ simple matrix with one column of each column sum and for any $t$) and not in $F_D(t)$ (for any $t$). Then $\text{forb}(m, F) = \Theta(m^k)$. 
Where could we go from here?
Linear algebra does work for the following case for which we already had an alternate proof.

\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

no configuration \( F = (t + 1) \cdot \begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix} \) forces
\[
\begin{pmatrix}
\leq t & \leq t & \leq t & \leq t \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
on any 4 rows. We can form a degree 2 indicator polynomial

\[
f(x) = x_1 x_2 - x_2 x_3 + x_3 x_4 - x_1 x_4
\]
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Linear algebra does work for the following case for which we already had an alternate proof.

\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\leq t \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

no configuration \( F = (t + 1) \cdot \begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} \) forces on any 4 rows. We can form a degree 2 indicator polynomial

\[
f(x) = x_1x_2 - x_2x_3 + x_3x_4 - x_1x_4
\]

**Theorem** Let \( t \geq 1 \) be given. Then \( \text{forb}(m, F) = \Theta(m^2) \).
Now if $A$ has no configuration $F = (t + 1) \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ this forces

\[
\begin{array}{cccc}
\leq t & \leq t & \leq t & \leq t \\
1 & 1 & 1 & 0 \\
\end{array}
\]

this forces

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

or

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

on any 4 rows. I have no idea how to use indicator polynomials for this case but we can conjecture a result.
Now if $A$ has no configuration $F = (t + 1) \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, this forces

\[
\begin{bmatrix}
\leq t & \leq t & \leq t & \leq t & \leq t & \leq t \\
1 & 1 & 1 & 0 & 0 & 0 \\
this forces & 1 & 0 & 0 & or & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

on any 4 rows. I have no idea how to use indicator polynomials for this case but we can conjecture a result.

**Conjecture** Let $t \geq 2$ be given. Then $\text{forb}(m, F)$ is $\Theta(m^2)$. 
Thanks for the invite to Banff!
Thanks for the invite to Banff!
Thanks to the organizers Chris Godsil, Peter Sin and Qing Xiang!
Thanks to all the participants for a wonderful conference!
The building blocks of our constructions are $I$, $I^c$ and $T$:

\[
I_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad I_4^c = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}, \quad T_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Note that

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix} \notin I, \quad \begin{bmatrix}
0 \\
0
\end{bmatrix} \notin I^c, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \notin T
\]
The building blocks of our constructions are $I$, $I^c$ and $T$:

\[
I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Note that

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T
\]

Note that $\text{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$
**Definition** Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\times
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
**Definition** Given an \( m_1 \times n_1 \) matrix \( A \) and a \( m_2 \times n_2 \) matrix \( B \) we define the product \( A \times B \) as the \( (m_1 + m_2) \times (n_1 n_2) \) matrix consisting of all \( n_1 n_2 \) possible columns formed from placing a column of \( A \) on top of a column of \( B \). If \( A, B \) are simple, then \( A \times B \) is simple. (A, Griggs, Sali 97)

\[
\begin{bmatrix}
1 & 0 & 0 \\
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0 & 0 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Given \( p \) simple matrices \( A_1, A_2, \ldots, A_p \), each of size \( m/p \times m/p \), the \( p \)-fold product \( A_1 \times A_2 \times \cdots \times A_p \) is a simple matrix of size \( m \times (m^p/p^p) \) i.e. \( \Theta(m^p) \) columns.
The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{l_{m/p}, l_{m/p}^c, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of $p$ such that $F$ is a configuration in every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{l_{m/p}, l_{m/p}^c, T_{m/p}\}$. 
**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of $p$ such that $F$ is a configuration in every $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$.

**Conjecture** (A, Sali 05) \textit{forb}(m, F) is $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.
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**Conjecture** (A, Sali 05) $\text{forb}(m, F)$ is $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

The conjecture has been verified for $k \times l \ F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.