Forbidden Configurations: A Survey

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October 23, 2012
I have worked with a number of coauthors in this area: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Christina Koch, Connor Meehan, U.S.R. Murty, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of \textit{VC-dimension} uses a forbidden configuration.

Survey at www.math.ubc.ca/∼anstee
An Elementary Extremal Problem

\[ [m] = \{1, 2, \ldots, m\} \]
\[ 2^{[m]} = \{A : A \subseteq [m]\} \]
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We say \( \mathcal{H} = ([m], \mathcal{E}) \) is a hypergraph if \( \mathcal{E} \subseteq 2^m \). The subsets in \( \mathcal{E} \) are called hyperedges.

**Theorem** If \( \mathcal{H} = ([m], \mathcal{E}) \) is a hypergraph, then

\[ |\mathcal{E}| \leq 2^m. \]
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**Theorem** If \( \mathcal{H} = ([m], \mathcal{E}) \) is a hypergraph, then

\[ |\mathcal{E}| \leq 2^m. \]

**Definition** We say a hypergraph \( \mathcal{E} \) is intersecting if for every pair \( A, B \in \mathcal{E} \), we have \( |A \cap B| \geq 1 \).

**Theorem** If \( \mathcal{H} = ([m], \mathcal{E}) \) is a hypergraph and \( \mathcal{E} \) is intersecting, then

\[ |\mathcal{E}| \leq 2^{m-1}. \]
Extremal Graph Theory often considers the following. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on $m$ vertices such that there is no subgraph $G$.

Let $\Delta$ denote the triangle on 3 vertices.

**Theorem** (Mantel 1907) $ex(m, \Delta) = |E(T(m, 2))| = \left\lfloor \frac{m^2}{4} \right\rfloor$
Extremal Graph Theory often considers the following. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on $m$ vertices such that there is no subgraph $G$.

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**Theorem** (Mantel 1907) $ex(m, \Delta) = |E(T(m, 2))| = \left\lfloor \frac{m^2}{4} \right\rfloor$

The Turán graph $T(m, k)$ on $m$ vertices are formed by partitioning $m$ vertices into $k$ nearly equal sets and joining any pair of vertices in different sets.
Let $\chi(G)$ denote the minimum number of colours required for the vertices of a graph $G$ so that no two adjacent vertices have the same colour. Thus $\chi(T(m, \ell)) = \ell$. If $\chi(G) = k$, then $G$ is not a subgraph of $T(m, k - 1)$, i.e. $\text{ex}(m, G) \geq |E(T(m, k - 1))|$. 

Theorem (Turán 41) Let $G$ denote the clique on $k$ vertices where every pair of vertices are joined. Then $\text{ex}(m, G) = |E(T(m, k - 1))|$. Note that $\chi(G) = k$.

Theorem (Erdős, Stone, Simonovits 46, 66) Let $G$ be a simple graph. Then $\lim_{m \to \infty} \frac{\text{ex}(m, G)}{m^2} = \lim_{m \to \infty} \frac{|E(T(m, \chi(G) - 1))|}{m^2} = 1 - \frac{1}{\chi(G) - 1}$.
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Note that \( \chi(G) = k \).

**Theorem** (Erdős, Stone, Simonovits 46, 66) Let \( G \) be a simple graph. Then

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\lim_{m \to \infty} \frac{\text{ex}(m, G)}{\binom{m}{2}} = \lim_{m \to \infty} \frac{|E(T(m, \chi(G) - 1))|}{\binom{m}{2}} = 1 - \frac{1}{\chi(G) - 1}.
\]
Note $\chi(\Delta) = 3$. If we consider a graph $H$ consisting of two disjoint copies of $\Delta$ then $\chi(H) = 3$ and $ex(m, H) > ex(m, \Delta)$. Yet

$$\lim_{m \to \infty} \frac{ex(m, H)}{\binom{m}{2}} = 1 - \frac{1}{3 - 1} = \frac{1}{2}.$$
We consider one possible generalization of the extremal problem: from graphs to hypergraphs, and subgraphs to subhypergraphs.
Consider a hypergraph $H$ with vertices $[4] = \{1, 2, 3, 4\}$ and with the following family of subsets as edges:
$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\} \subseteq 2^{[4]}$$

The incidence matrix $A$ of the family $\mathcal{A} \subseteq 2^{[4]}$ is:

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
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\end{bmatrix}$$

**Definition** We say that a matrix $A$ is *simple* if it is a $(0,1)$-matrix with no repeated columns.

**Definition** We define $\|A\|$ to be the number of columns in $A$.

$$\|A\| = 6 = |\mathcal{A}|$$
Definition Given a matrix $F$, we say that $A$ has $F$ as a configuration if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$
We consider the property of forbidding a configuration $F$ in $A$.

**Definition** Let

$$\text{forb}(m, F) = \max\{ \|A\| : A \text{ is a } m\text{-rowed simple, no configuration } F \}$$
We consider the property of forbidding a configuration $F$ in $A$.

**Definition** Let

$$forb(m, F) = \max\{\|A\| : A \text{ $m$-rowed simple, no configuration } F\}$$

E.g. $forb(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$
**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$
Some Main Results

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

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When a matrix $A$ has a copy of $K_k$ on some $k$-set of rows $S$, then we say that $A$ **shatters** $S$. It is useful to define $A$ to have **VC-dimension** $k$ if $k$ is the maximum cardinality of a shattered set in $A$.

Thomassé argued that this is the hypergraph version of tree width.
One result about VC-dimension involves the following. Let $S \subset [m]$ be a transversal of $A$ if each column of $A$ has at least one 1 in a row of $S$. Seeking a minimum transversal, we let $x$ be the $(0,1)$-incidence vector of $S$, and compute:

$$
\tau = \min \left\{ 1 \cdot x \text{ subject to } A^T x \geq \mathbf{1}, \quad x \in \{0, 1\}^m \right\}.
$$

The natural fractional problem is:

$$
\tau^* = \min \left\{ 1 \cdot x \text{ subject to } A^T x \geq \mathbf{1}, \quad x \geq 0 \right\}.
$$

Theorem (Haussler and Welzl 87) If $A$ has VC-dimension $k$ then

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\tau \leq 16 k \tau^* \log(k \tau^*).
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**Theorem** (Haussler and Welzl 87) If $A$ has VC-dimension $k$ then $\tau \leq 16k\tau^* \log(k\tau^*)$. 
Let $sh(A) = \{ S \subseteq [m] : A \text{ shatters } S \}$

e.g.

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
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\end{bmatrix}$$

$$sh(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}\}$$
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$$A = \begin{bmatrix}
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\( sh(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\} \} \)

\( 6 = \| A \| \leq |sh(A)| = 7 \)
Let \( sh(A) = \{ S \subseteq [m] : A \text{ shatters } S \} \)

**Theorem** (Pajor 85) \( \|A\| \leq |sh(A)|. \)

**Proof:** Decompose \( A \) as follows:

\[
A = \begin{bmatrix}
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\[\|A\| = \|A_0\| + \|A_1\|.\]
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**Proof:** Decompose \( A \) as follows:

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A = \begin{bmatrix}
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A_0 & A_1 
\end{bmatrix}
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\( \|A\| = \|A_0\| + \|A_1\| \).

By induction \( \|A_0\| \leq |sh(A_0)| \) and \( \|A_1\| \leq |sh(A_1)| \).
Let $sh(A) = \{ S \subseteq [m] : A \text{ shatters } S \}$

**Theorem** (Pajor 85) \[ \|A\| \leq |sh(A)|. \]

**Proof:** Decompose $A$ as follows:

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & 11 & \cdots & 1 \\
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\end{bmatrix}
\]

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By induction $\|A_0\| \leq |sh(A_0)|$ and $\|A_1\| \leq |sh(A_1)|$.

\[ |sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)| \]
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If $S \in sh(A_0) \cap sh(A_1)$, then $1 \cup S \in sh(A)$. 
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If \( S \in sh(A_0) \cap sh(A_1) \), then \( 1 \cup S \in sh(A) \).

So \( (sh(A_0) \cup sh(A_1)) \cup (1 + (sh(A_0) \cap sh(A_1))) \subseteq sh(A) \).
Let $sh(A) = \{ S \subseteq [m] : A \text{ shatters } S \}$

**Theorem** (Pajor 85) $\|A\| \leq |sh(A)|$.

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\|A\| = \|A_0\| + \|A_1\|.
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|sh(A_0) \cup sh(A_1)| = |sh(A_0)| + |sh(A_1)| - |sh(A_0) \cap sh(A_1)|
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If \( S \in sh(A_0) \cap sh(A_1) \), then \( 1 \cup S \in sh(A). \)

So \( (sh(A_0) \cup sh(A_1)) \cup (1 + (sh(A_0) \cap sh(A_1))) \subseteq sh(A). \)

\[
|sh(A_0)| + |sh(A_1)| \leq |sh(A)|.
\]

Hence \( \|A\| \leq |sh(A)|. \)
**Remark** If $A$ shatters $S$ then $A$ shatters any subset of $S$.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

**Proof:** Let $A$ have no $K_k$. 
Remark If $A$ shatters $S$ then $A$ shatters any subset of $S$.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

\[
\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}
\]

**Proof:** Let $A$ have no $K_k$.

Then $sh(A)$ can only contain sets of size $k - 1$ or smaller.

Then

\[
||A|| \leq |sh(A)| \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.
\]
Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

\[
\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \quad \text{which is } \Theta(m^{k-1}).
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Some Main Results

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**Corollary** Let \( F \) be a \( k \times \ell \) simple matrix. Then \( \text{forb}(m, F) = O(m^{k-1}). \)
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**Theorem** (Füredi 83). Let \( F \) be a \( k \times \ell \) matrix. Then
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Some Main Results

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**Theorem** (Füredi 83). Let \( F \) be a \( k \times \ell \) matrix. Then

\[forb(m, F) = O(m^k).\]

**Problem** Given \( F \), can we predict the behaviour of \( forb(m, F) \)?
**Definition** Let $F$ be a $k$-rowed configuration and let $\alpha$ be a $k$-rowed column vector. Define $[F|\alpha]$ to be the concatenation of $F$ and $\alpha$.

**Theorem** (A, Fleming 10) Let $F$ be a $k \times \ell$ simple matrix satisfying . . . various conditions. . . Then $\text{forb}(m, F)$ is $O(m^{k-2})$ (instead of $\Theta(m^{k-1})$). Moreover if $F$ satisfies . . . various additional conditions. . . then for any $k$-rowed column $\alpha$ not in $F$, we have $\text{forb}(m, [F|\alpha])$ is $\Theta(m^{k-1})$. 
A result for simple $F$

Let $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**Theorem** (A, Fleming 10) Let $F$ be a $k \times \ell$ simple matrix such that there is a pair of rows with no configuration $E_1$ and there is a pair of rows with no configuration $E_2$ and there is a pair of rows with no configuration $E_3$. Then $\text{forb}(m, F)$ is $O(m^{k-2})$. 
Let \( E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \)

**Theorem** (A, Fleming 10) Let \( F \) be a \( k \times \ell \) simple matrix such that there is a pair of rows with no configuration \( E_1 \) and there is a pair of rows with no configuration \( E_2 \) and there is a pair of rows with no configuration \( E_3 \). Then \( \text{forb}(m, F) \) is \( O(m^{k-2}) \).

Moreover if we select three subsets \( I_1, I_2, I_3 \subseteq [k] \) with

\(|I_1| = |I_2| = |I_3| = 2\) and \( F \) is the column maximal \( k \)-rowed simple matrix that has no configuration \( E_1 \) on rows \( I_1 \), no configuration \( E_2 \) on rows \( I_2 \) and no configuration \( E_3 \) on rows \( I_3 \), then for any \( k \)-rowed column \( \alpha \) not in \( F' \), we have \( \text{forb}(m, [F|\alpha]) \) is \( \Theta(m^{k-1}) \).
Example

\[ F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad forb(m, F) = 2m \]
The building blocks of our product constructions are $I$, $I^c$ and $T$, e.g:

\[
I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
The Conjecture

We conjecture that product constructions with the three building blocks \( \{I, I^c, T\} \) determine the asymptotically best constructions.
**Definition** Given two matrices $A, B$, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of $A$ on top of a column of $B$ in all possible ways. (A, Griggs, Sali 97)

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.
Examples

\[
[01] \times [01] = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\end{bmatrix} = K_2
\]

\(I_{m/2} \times I_{m/2}\) is vertex-edge incidence matrix of \(K_{m/2,m/2} = T(m,2)\)
The Conjecture \( \chi(G) \rightarrow x(F) \)?

We conjecture that product constructions with the three building blocks \( \{I, I^c, T\} \) determine the asymptotically best constructions.

**Definition** Let \( F \) be given. Let \( x(F) \) denote the largest \( p \) such that there is a \( p \)-fold product which does not contain \( F \) as a configuration where the \( p \)-fold product is \( A_1 \times A_2 \times \cdots \times A_p \) where each \( A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\} \).

**Conjecture** (A, Sali 05) \( \text{forb}(m, F) \) is \( \Theta(m^{x(F)}) \).
We conjecture that product constructions with the three building blocks \( \{I, I^c, T\} \) determine the asymptotically best constructions.

**Definition** Let \( F \) be given. Let \( x(F) \) denote the largest \( p \) such that there is a \( p \)-fold product which does not contain \( F \) as a configuration where the \( p \)-fold product is \( A_1 \times A_2 \times \cdots \times A_p \) where each \( A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\} \).

**Conjecture** (A, Sali 05) \( \text{forb}(m, F) \) is \( \Theta(m^{x(F)}) \).

The conjecture has been verified for \( k \times \ell F \) where \( k = 2 \) (A, Griggs, Sali 97) and \( k = 3 \) (A, Sali 05) and \( \ell = 2 \) (A, Keevash 06) and for \( k \)-rowed \( F \) with bounds \( \Theta(m^{k-1}) \) or \( \Theta(m^k) \) (A, Fleming 11) plus other cases.
6-rowed Configurations with Quadratic Bounds

\[ G_{6 \times 3} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \]

**Theorem** (A, Raggi, Sali 11) \( \text{forb}(m, G_{6 \times 3}) \) is \( \Theta(m^2) \). Moreover \( G_{6 \times 3} \) is a boundary case, namely for any column \( \alpha \), then \( \text{forb}(m, [G_{6 \times 3} | \alpha]) \) is \( \Omega(m^3) \). In fact if \( F \) is 6-rowed and not a configuration in \( G_{6 \times 3} \), then \( \text{forb}(m, F) \) is \( \Omega(m^3) \).

The proof uses induction to reduce to a 5-rowed case which is then established by a quite complicated induction.
**Definition** Let \( t \cdot M \) be the matrix \([M | M | \cdots | M]\) consisting of \( t \) copies of \( M \) placed side by side.
**Definition** Let $t \cdot M$ be the matrix $[M | M | \cdots | M]$ consisting of $t$ copies of $M$ placed side by side.

**Theorem** (A, Füredi 86) Let $t, k$ be given.

$$forb(m, t \cdot K_k) \leq t - 2 \left( \frac{m}{k} \right) + \left( \frac{m}{k} \right) + \left( \frac{m}{k-1} \right) + \cdots + \left( \frac{m}{0} \right)$$

with equality if a certain $k$-design exists.
A 4-rowed \( F \) with a quadratic bound

Using the result of A and Fleming 10, there are three simple column-maximal 4-rowed \( F \) for which \( \text{forb}(m, F) \) is quadratic. Here is one example:

\[
F_8 = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

Can we repeat columns in \( F_8 \) and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.
A 4-rowed $F$ with a quadratic bound

Using the result of A and Fleming 10, there are three simple column-maximal 4-rowed $F$ for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Can we repeat columns in $F_8$ and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed $t$, let

$$F_8(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
A 4-rowed $F$ with a quadratic bound

$$F_8(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

**Theorem** (A, Raggi, Sali 09) Let $t$ be given. Then $\text{forb}(m,F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column $\alpha$ not already present $t$ times in $F_8(t)$, then $\text{forb}(m,[F_8(t)|\alpha])$ is $\Omega(m^3)$.

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.
We have been able to determine an ‘easy’ criteria for $k$-rowed $F$ for which $\text{forb}(m, F)$ is $O(m^{k-1})$ as opposed to $\Theta(m^k)$.

**Theorem** (A., Fleming, Sali, Füredi 05, A., Fleming 11)
There is an . . . easy to describe list . . . of various $k$-rowed $F$ for which $\text{forb}(m, F)$ is $O(m^{k-1})$.

Moreover if a $k$-rowed $F$ is not a configuration in one of the . . . easy to describe list . . . then $\text{forb}(m, F)$ is $\Theta(m^k)$. 
We have been able to determine an ‘easy’ criteria for \( k \)-rowed \( F \) for which \( \text{forb}(m, F) \) is \( O(m^{k-1}) \) as opposed to \( \Theta(m^k) \).

**Theorem** (A., Fleming, Sali, Füredi 05, A., Fleming 11)

Let \( D(k) \) denote the matrix with all columns of sum at least 1 except those columns with 1’s on both rows 1 and 2. Then \( \text{forb}(m, [0_k \mid t \cdot D(k)]) \) is \( O(m^{k-1}) \).

Let \( B \) be an \( k \times (k+1) \) matrix with one column of each column sum. Then \( \text{forb}(m, [K_k \mid t \cdot [K_k \backslash B]]) \) is \( O(m^{k-1}) \).

Moreover if \( F \) is a \( k \)-rowed configuration not a configuration in either \( [0 \mid t \cdot D(k)] \) or in \( [K_k \mid t \cdot [K_k \backslash B]] \), for some \( k \times (k+1) \) matrix \( B \) with one column of each column sum, then \( \text{forb}(m, F) \) is \( \Theta(m^k) \).
Determining exact bounds $forb(m, F)$ can be very challenging requiring much more understanding than is required for asymptotic bounds. The proofs often depend heavily on the structures of $m \times forb(m, F)$ simple matrices that avoid $F$; these are called extremal matrices. Despite the challenge (and fun!) of obtaining exact bounds, I suspect the asymptotic results are more important.
**Theorem** (A., Kamoosi 07)

\[ \text{forb}(m, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}) = \left\lfloor \frac{10}{3}m - \frac{4}{3} \right\rfloor. \]
Exact Bounds

**Theorem** (A., Karp 10)

\[
forb(m, \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}) = \frac{4}{3} \binom{m}{2} + m + 1
\]

for \(m \equiv 1, 3(\text{mod } 6)\).

A simple design theoretic construction (Steiner Triple System) yields the lower bound \(forb(m, \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}) \geq \frac{4}{3} \binom{m}{2} + m + 1\) while a pigeonhole argument yields the upper bound \(forb(m, \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}) \leq \frac{4}{3} \binom{m}{2} + m + 1\). Extending this upper bound to the \(3 \times 3\) matrix requires a careful matching argument.
Definition Let $K^\ell_{k}$ denote the $k \times \binom{k}{\ell}$ simple matrix of all possible columns of sum $\ell$ on $k$ rows.

Theorem (A., Barekat) Let $q$ be given and let $F$ be the following $4 \times q$ matrix

\[
F = \begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Then

\[
forb(m, F) \leq \binom{m}{0} + \binom{m}{1} + \frac{q + 3}{3} \binom{m}{2} + \binom{m}{m-1} + \binom{m}{m}
\]

with equality if $m \geq q$ and $m \equiv 1, 3 \pmod{6}$ and

\[
A = [K^0_m K^1_m K^2_m T_{m,a} T^c_{m,b} K^{m-2}_m K^{m-1}_m K^m_m]
\]

(for some choice $a, b$ with $a + b = q - 3$).
The following result, found with the help of Genetic algorithms, has a bound just 2 larger than the one we initially expected.

**Theorem** (A., Raggi 11) Assume \( m \geq 6 \). Then
\[
\text{forb}(m, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}) = \binom{m}{2} + m + 4.
\]

The construction was relatively complicated:

\[
A = \left[ \begin{array}{ccccccccccc}
U & 0_u & 0_u & 0_u & 0_u & K^1_u & 1_u & K^1_u & K^2_u & 1_u \\
L & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0_\ell & 1_\ell & K^l-1_\ell & K^l-2_\ell & K^l-1_\ell & 0_\ell & 1_\ell & 1_\ell & 1_\ell & 1_\ell
\end{array} \right].
\]
Critical Substructures

**Definition** A *critical substructure* of a configuration $F$ is a minimal configuration $F'$ contained in $F$ such that

$$\text{forb}(m, F') = \text{forb}(m, F).$$

A critical substructure $F'$ has an associated construction avoiding it that yields a lower bound on $\text{forb}(m, F')$. Some other argument provides the upper bound for $\text{forb}(m, F)$. When $F'$ is a configuration in $F''$ and $F''$ is a configuration in $F$, we deduce that

$$\text{forb}(m, F') = \text{forb}(m, F'') = \text{forb}(m, F).$$

Let $1_{k0_\ell}$ denote the $(k + \ell) \times 1$ column of $k$ 1’s on top of $\ell$ 0’s.
Critical Substructures for $K_4$

$$K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$.

Note that $forb(m, 1_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1) = forb(m, 0_4) = forb(m, 2 \cdot 1_3) = forb(m, 2 \cdot 0_3)$. 
Critical Substructures for $K_4$

$K_4 = \begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}$

Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$.
Note that $\text{forb}(m, 1_4) = \text{forb}(m, K_4^3) = \text{forb}(m, K_4^2) = \text{forb}(m, K_4^1) = \text{forb}(m, 0_4) = \text{forb}(m, 2 \cdot 1_3) = \text{forb}(m, 2 \cdot 0_3)$. 
Critical Substructures for $K_4$

$$K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
\end{bmatrix}$$

Critical substructures are $1_4, K_4^3, K_4^2, K_4^1, 0_4, 2 \cdot 1_3, 2 \cdot 0_3$.

Note that $forb(m, 1_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1) = forb(m, 0_4) = forb(m, 2 \cdot 1_3) = forb(m, 2 \cdot 0_3)$. 
Critical Substructures for $K_4$

Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$.

Note that $forb(m, 1_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1) = forb(m, 0_4) = forb(m, 2 \cdot 1_3) = forb(m, 2 \cdot 0_3)$.  

\[
K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Critical Substructures for $K_4$

$$K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}$$

Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$. Note that $\text{forb}(m, 1_4) = \text{forb}(m, K_4^3) = \text{forb}(m, K_4^2) = \text{forb}(m, K_4^1) = \text{forb}(m, 0_4) = \text{forb}(m, 2 \cdot 1_3) = \text{forb}(m, 2 \cdot 0_3)$. 
Critical Substructures for \( K_4 \)

\[
K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Critical substructures are \( 1_4, K_4^3, K_4^2, K_4^1, 0_4, 2 \cdot 1_3, 2 \cdot 0_3 \).

Note that \( forb(m, 1_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1) = forb(m, 0_4) = forb(m, 2 \cdot 1_3) = forb(m, 2 \cdot 0_3) \).
Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$. Note that $\text{forb}(m, 1_4) = \text{forb}(m, K_4^3) = \text{forb}(m, K_4^2) = \text{forb}(m, K_4^1) = \text{forb}(m, 0_4) = \text{forb}(m, 2 \cdot 1_3) = \text{forb}(m, 2 \cdot 0_3)$. 

Critical Substructures for $K_4$ 

$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$
Critical Substructures for $K_4$

$$K_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

Critical substructures are $1_4$, $K_4^3$, $K_4^2$, $K_4^1$, $0_4$, $2 \cdot 1_3$, $2 \cdot 0_3$. Note that $forb(m, 1_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1) = forb(m, 0_4) = forb(m, 2 \cdot 1_3) = forb(m, 2 \cdot 0_3)$. 
We can extend $K_4$ and yet have the same bound

$$[K_4|1_20_2] =$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

**Theorem** (A., Meehan) For $m \geq 5$, we have

$$\text{forb}(m, [K_4|1_20_2]) = \text{forb}(m, K_4).$$
We can extend $K_4$ and yet have the same bound

$$[K_4 | 1_2 0_2] =$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

**Theorem** (A., Meehan) For $m \geq 5$, we have

$$\text{forb}(m, [K_4 | 1_2 0_2]) = \text{forb}(m, K_4).$$

We expect in fact that we could add many copies of the column $1_2 0_2$ and obtain the same bound, albeit for larger values of $m$. 
\[
F_{2110} = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Not all \( F \) are likely to yield simple exact bounds:

**Theorem** Let \( c \) be a positive real number. Let \( A \) be an \( m \times \left( c\binom{m}{2} + m + 2 \right) \) simple matrix with no \( F_{2110} \). Then for some \( M > m \), there is an
\[
M \times \left( c + \frac{2}{m(m-1)} \binom{M}{2} + M + 2 \right) \] simple matrix with no \( F_{2110} \).

**Theorem** (P. Dukes 09) \( \text{forb}(m, F_{2110}) \leq .691m^2 \)

The proof used inequalities and linear programming.
Determine the asymptotics for $\text{forb}(m, F)$ for

$$F = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.$$  

(Hard!)
Open Problems

Determine an exact bound for $\text{forb}(m, F)$ for

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
Show that $2 \cdot 1_4$ (the $4 \times 2$ matrix of 1’s) and $2 \cdot 0_4$ are the only 4-rowed critical substructures of $K_5$ by showing that for $m$ large,

$$\text{forb}(m, [0_4 | 2 \cdot K^1_4 | 2 \cdot K^2_4 | 2 \cdot K^3_4 | 1_4]) < \text{forb}(m, K_5).$$
THANKS to Andre for arranging this seminar!
Theorem (A, Raggi, Sali) \( \text{forb}(m, F_7) \) is \( \Theta(m^2) \). Moreover \( F_7 \) is a boundary case, namely for any column \( \alpha \), then \( \text{forb}(m, [F_7|\alpha]) \) is \( \Omega(m^3) \).

The Conjecture predicts nine 5-rowed simple matrices \( F \) to be boundary cases of which this is one.
Induction

Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \left[ \begin{array}{cccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{array} \right].$$
Induction

Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{bmatrix}.$$

Now $[B_r C_r D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$. 
$C_r$ has no $F$ in

$$
\mathcal{F} = \begin{\left\{ \begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \right. \begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1
\end{bmatrix} \right\}
$$
Induction

Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{l}
\text{row } r \\
0 \cdots 0 1 \cdots 1 \\
B_r \quad C_r \quad C_r \quad D_r
\end{array}.$$ 

Now $[B_r C_r D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$. 
Let $A$ be an $m \times \text{forb}(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{bmatrix}.$$ 

Now $[B_r C_r D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$. Then

$$\|A\| = \text{forb}(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq \text{forb}(m - 1, F_7) + \|C_r\|.$$
Let $A$ be an $m \times forb(m, F_7)$ simple matrix with no configuration $F_7$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{array} \right].$$

Now $[B_r \, C_r \, D_r]$ is an $(m - 1)$-rowed simple matrix with no configuration $F_7$. Also $C_r$ is an $(m - 1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from $F_7$. Then

$$\|A\| = forb(m, F_7) = \|B_r \, C_r \, D_r\| + \|C_r\| \leq forb(m - 1, F_7) + \|C_r\|.$$

To show $forb(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of $r$. 

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Forbidden Configurations: A Survey
Let $C_r$ be an $(m - 1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s_i$ and reorder rows and columns to obtain

$$C_r = \text{row } s_i \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & G_i & G_i & H_i \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of $s_i$. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in a contradiction.
Let $C_r$ be an $(m - 1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row $s_i$ and reorder rows and columns to obtain

$$
C_r = \text{row } s_i \begin{bmatrix}
0 & \cdots & 0 & 1 & \cdots & 1 \\
E_i & G_i & G_i & H_i
\end{bmatrix}.
$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of $s_i$. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in a contradiction.

Idea: Select a minimal set of rows $L_i$ so that $G_i|L_i$ is simple.
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$$C_r = \text{row } s_i \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & G_i & G_i & H_i \\ \text{columns } \subseteq [0|I] \end{array} \right] \{L_i\}.$$
Idea: Select a minimal set of rows $L_i$ so that $G_i|_{L_i}$ is simple.

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Idea: Select a minimal set of rows $L_i$ so that $G_i|_{L_i}$ is simple.

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We may choose $s_1$ and form $L_1$.  
Then choose $s_2 \in L_1$ and form $L_2$.  
Then choose $s_3 \in L_2$ and form $L_3$.  

etc.  

We can show the sets $L_1 \setminus s_2, L_2 \setminus s_3, L_3 \setminus s_4, \ldots$ are disjoint.  
Assuming $\|G_i\| \geq 8$ for all choices $s_i$ results in $|L_i \setminus s_{i+1}| \geq 3$ which yields a contradiction.