# A Survey of Forbidden Configuration Results (Draft)

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**Abstract**

This paper surveys various results concerning forbidden configurations that have been obtained by Aldred, Anstee, Barekat, Chervonenkis, Dunwoody, Farber, Ferguson, Fleming, Frankl, Füredi, Griggs, Gronau, Kamoosi, Karp, Keevash, Murty, Pach, Perles, Quinn, Ryan, Sali, Sauer, Shelah, and Vapnik to name a few.

Let $F$ be a $k \times \ell$ (0,1)-matrix (the forbidden configuration). We define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Assume $m$ is given and assume $A$ is an $m \times n$ simple matrix which has no submatrix which is a row and column permutation of $F$. We define $\mathrm{forb}(m, F)$ as the best possible upper bound on $n$ depending on $m$ and $F$. We seek exact values for $\mathrm{forb}(m, F)$ as well as seeking asymptotic results for $\mathrm{forb}(m, F)$ for a fixed $F$ and as $m$ tends to infinity. A conjecture of Anstee and Sali predicts the asymptotically best constructions from which to derive the asymptotics of $\mathrm{forb}(m, F)$.

Keywords: forbidden configurations, extremal set theory, (0,1)-matrices, trace.

## 1 Introduction

The study of forbidden configurations is a problem in extremal set theory. It is convenient to use the language of matrix theory. We define a *simple* matrix as a (0,1)-matrix with no repeated columns. Such an $m \times n$ simple matrix can be thought of a family of $n$ subsets of $\{1, 2, \ldots, m\}$ with the rows indexing the elements and the columns indexing

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the subsets. Assume we are given a $k \times \ell$ (0,1)-matrix $F$. We say that a matrix $A$ has a *configuration* $F$ if a submatrix of $A$ is a row and column permutation of $F$ and so $F$ is referred to as a *configuration* of $A$ (sometimes called *trace*).

The reader may ask of the importance of the configuration idea in combinatorial investigations. I feel it is one of a few possible basic notions of substructure and it has arisen in applications.

We define $\text{forb}(m, F)$ as the smallest value (depending on $m$ and $F$) so that if $A$ is a simple $m \times n$ matrix and $A$ has no configuration $F$ then $n \le \text{forb}(m, F)$. Alternatively $\text{forb}(m, F)$ is the smallest value so that if $A$ is an $m \times (\text{forb}(m, F)+1)$ simple matrix then $A$ must have a configuration $F$. We are focusing on a single fixed forbidden configuration (though variations are in Section 14) as we let $m$ grow.

We often blur the distinction between a matrix $F$ and the related equivalence class $\tilde{F}$ of matrices under arbitrary row and column permutations. We can say that a matrix $A$ has a configuration $F$ if $A$ has a submatrix in $\tilde{F}$. A matrix $F$ is referred to as a *configuration* when we wish to consider whether another matrix $A$ has $F$ as a configuration.

We use the notation $[A|B]$ to denote the matrix obtained from concatenating the two matrices $A$ and $B$. We use the notation $k \cdot A$ to denote the matrix $[A|A|\cdots|A]$ consisting of $k$ copies of $A$ concatenated together. We give precedence to the operation $\cdot$ (multiplication) over concatenation so that for example $[2 \cdot A|B]$ is the matrix consisting of the concatenation of $B$ with the concatenation of two copies of $A$. We make a few simple observations.

**Remark 1.1** *If we let $A^c$ denote the 0-1-complement of $A$ then $forb(m, F^c) = forb(m, F)$.*

**Remark 1.2** *If $F'$ is a row and column permutation of a submatrix of $F$ (i.e. $F$ has a configuration $F'$), then $forb(m, F') \le forb(m, F)$.*

When giving results it is often convenient to note when we have $\text{forb}(m, F') = \text{forb}(m, F)$ where $F'$ is a configuration in $F$. Typically one has a construction working for $F'$ (a simple matrix $A$ with no configuration $F'$) which then necessarily works for $F$ and we have a bound for $\text{forb}(m, F)$ which certainly applies to $\text{forb}(m, F')$. Equality (or asymptotic equality) of the construction and the bound then yields equality (or asymptotic equality) for $\text{forb}(m, F')$ and $\text{forb}(m, F)$ as well as any matrices intermediate between $F'$ and $F$.

Some notations help us describe the most important matrices. Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible (0,1)-columns on $k$ rows and let $K_k^s$ denote the $k \times \binom{k}{s}$ simple matrix of all possible columns of column sum $s$. Many results have been obtained about $\text{forb}(m, F)$ but the following is the most fundamental.

**Theorem 1.3** *[Sauer [47], Perles and Shelah [48], Vapnik and Chervonenkis [49]] We have that*

$$forb(m, K_k) = forb(m, K_k^k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

*and so $forb(m, K_k)$ is $\Theta(m^{k-1})$.*

There is an easy induction proof (Section 9), a shifting proof (Section 10), a linear algebra proof (Section 12) of Theorem 1.3. The asymptotic growth of $\Theta(m^{k-1})$ was what interested Vapnik and Chervonenkis. An easy consequence of Theorem 1.3 using Remark 1.2 is the following

**Corollary 1.4** *Let $F$ be a $k \times \ell$ simple matrix. Then forb$(m, F)$ is $O(m^{k-1})$.* ■

It would seem reasonable to consider (0,1)-matrices $F$ which are not simple as well. Füredi [39] noted the following general bound that can be proved using Theorem 1.3.

**Theorem 1.5** *[39] Let $F$ be a $k \times \ell$ (0,1)-matrix. Then there is a constant $c_F$ so that forb$(m, F) \leq c_f m^k$ i.e. forb$(m, F)$ is $O(m^k)$.*

There are some quite general results. The first result (simultaneously and independently obtained by Füredi and Quinn (generalizing a result of Ryser[46]) and the second result of Gronau are both exact and can be deduced by the existence of constructions since the bounds follows from Remark 1.2 in the first case using $F = K_k$ and in the second case using $F = K_{k+1}$.

**Theorem 1.6** *[40] Let $k, s$ be given positive integers with $0 \leq s \leq k$.*

$$\mathit{forb}(m, K_k^s) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}. \blacksquare$$

**Theorem 1.7** *[41]*

$$\mathit{forb}(m, 2 \cdot K_k) = \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}. \blacksquare$$

**Theorem 1.8** *[23]*

$$\mathit{forb}(m, t \cdot K_k) = \mathit{forb}(m, t \cdot K_k^k) = \frac{t-2}{k+1}\binom{m}{k}(1-o(1)) + \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}. \blacksquare$$

This latter result is another view of Füredi's result Theorem 1.5. An exact bound is only available after solving forb$(m, t \cdot K_k^k)$ which is essentially a design type problem. The following two results are quite general refinements of Theorems 1.3 and 1.5. The following describes the boundary between $\Theta(m^{k-2})$ and $\Theta(m^{k-1})$ for simple $k \times \ell$ $F$.

**Theorem 1.9** *[20]*

*If $F$ is a simple $k \times \ell$ matrix with the property that there is a pair of rows of $F$ that do not contain $K_2^0$, a pair of rows of $F$ that do not contain $K_2^2$ and a pair of rows of $F$ that do not contain the configuration $K_2^1 = I_2$, then forb$(m, F)$ is $O(m^{k-2})$.*

*If $F$ is a simple $k \times \ell$ matrix with the property that either every pair of rows has $K_2^0$ or every pair of rows has $K_2^2$ or every pair of rows has $K_2^1$, then forb$(m, F)$ is $\Theta(m^{k-1})$.*

The following considers the boundary between $\Theta(m^{k-1})$ and $\Theta(m^k)$ for arbitrary $k \times \ell$ $F$. The result in Theorem 1.10 were first proved for $k = 3$ in [32],[24] (there were two proofs originally, one for each of the two possible choices of a $3 \times 4$ $B$) and Theorem 1.11 were first proved for $k = 3$ in [32]. Theorem 1.10 was proven for general $k$ in [20],[22] and Theorem 1.11 was proven for general $k$ in [21].

**Theorem 1.10** *[24][22][32] Let $B$ be a simple $k \times k+1$ matrix with the property that there is one column of each column sum. Let $K_k - B$ denote the $k \times (2^k - k - 1)$ matrix obtained from $K_k$ by deleting the columns of $B$ (row order matters here). Let $t$ be given. Then $forb(m, [K_k \,|\, t \cdot [K_k - B]])$ is $\Theta(m^{k-1})$.* ■

**Theorem 1.11** *[32][21] Let $k$ be given and let $D_{12}$ denote the simple matrix of all columns of column sum at least 1 with no $K_2^2$ on rows 1 and 2. Then assuming $k \geq 3$ and $t \geq 2$ then $forb(m, [K_k^0 \,|\, t \cdot [2 \cdot K_k^1\, D_{12}])$ is $\Theta(m^{k-1})$.* ■

**Theorem 1.12** *[21] Let $F$ be a $k$-rowed matrix with maximum column multiplicity $t$. If $F$ is not a configuration in $forb(m, [K_k \,|\, (t-1) \cdot [K_k - B]])$ for some choice of $B$ as in Theorem 1.10 and $F$ is not a configuration in $forb(m, [K_k^0 \,|\, t \cdot [K_k^1\, D_{12}])$ for $D_{12}$ as in Theorem 1.11 then $forb(m, F)$ is $\Theta(m^k)$.*

This completely determines the boundary between $\Theta(m^k)$ and $\Theta(m^{k-1})$.

A large number of exact bounds are sprinkled throughout this survey including complete exact results for $1 \times \ell$ $F$ in Section 3 and complete exact results for $k \times 1$ $F$ in Section 6, a number of $2 \times \ell$ results in Section 3 and a number of $k \times 2$ results in Section 6 as well as a number of $3 \times 3$ and $3 \times 4$ results in Section 4. Exact bounds often require a more complete understanding of what it means to forbid a configuration and for example it is sometimes possible to classify the extremal matrices.

The purpose of this paper is to summarize existing results (Sections 3, 4, 5, 6, 7) and the proof techniques employed (Sections 8, 9, 10, 12, 11, 13). In doing so, we are encouraging the gentle reader to consider ways to make progress in proving the conjecture described in Section 2 or perhaps obtaining exact bounds or exploring other related problems such as described in Section 14. Open problems are scattered throughout including Conjecture 2.2, Problem 2.4, Problem 5.4, Problem 6.5, Conjecture 7.1, Problem 13.2, Conjecture 14.10. Here are two problems that I can suggest:

**Problem 1.13** *Show that*

$$forb(m, \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}) \text{ is } O(m^2).$$

**Problem 1.14** *Show that*

$$forb(m],)[= \frac{5}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}.$$

I expect that I have missed many related results that have been stated in another context but have relevance here. I would be glad to hear about them; email me at anstee@math.ubc.ca. There are some alternate ways of expressing the forbidden configuration problem that should be considered. The direct use of set notation rather than matrix notation is sometimes preferable. Some useful notation is

$$[m] = \{1, 2, \ldots, m\}, \quad 2^{[m]} = \{A \subseteq [m] : 0 \le |A| \le m\}, \quad \binom{[m]}{k} = \{A \subseteq [m] : |A| = k\}.$$

In this way $K_k$ corresponds to $2^{[k]}$ and $K_k^s$ corresponds to $\binom{[k]}{s}$. Our simple $m \times n$ matrix $A$ an be thought of as a family of sets

$$\mathcal{F} \subseteq 2^{[m]}, \qquad |\mathcal{F}| = n$$

and a configuration (if simple) corresponds to the trace. A *k-uniform* set system $\mathcal{F}$ has $\mathcal{F} \subseteq \binom{[m]}{k}$.

Another equivalent notation is to consider a square free integer $x = \prod_{i=1}^{m} p_i$ and then consider all possible divisors of $x$. This notation was used in [1] and in that case was generalized to all divisors of some given but arbitrary integer. See this multiset version in Section 14.

# 2 A Conjecture for asymptotic bounds

Our investigations have led us to a conjecture on the asymptotic growth of $\text{forb}(m, F)$ for a fixed $F$ as $m$ goes to infinity. We had noted early on that all our results had $\text{forb}(m, F) = \Theta(m^e)$ for an integer $e$. Our conjecture involves a cross product construction. Let $A_i$ be an $m_i \times n_i$ simple matrix for $1 \le i \le t$. Denote the $t$-fold product $A_1 \times A_2 \times \cdots \times A_t$ as the $(\sum m_i) \times (\Pi n_i)$ simple matrix whose columns are formed in all possible ways by putting a column of $A_1$ in the first $m_1$ rows and putting a column of $A_2$ in the next $m_2$ rows etc. If we allow row and column permutations then $A_1 \times A_2$ is the same as $A_2 \times A_1$. Let $I_h$ denote the $h \times h$ identity matrix and $I_h^c$ denotes its (0,1)-complement. Let $T_h$ denote the $h \times h$ *triangular* matrix

$$T_h = \begin{bmatrix} 1 & & & 1's \\ & 1 & & \\ & & \ddots & \\ 0's & & & 1 \end{bmatrix}.$$

The three matrices $I, I^c, T$ are our proposed building blocks for the product construction. Note that if each $A_i$ in the $t$-fold product above is of size $m/t \times m/t$ then the $t$-fold product has $m$ rows and $\Theta(m^t)$ columns. Let $F$ be a $k \times \ell$ (0,1)-matrix.

**Definition 2.1** *Let $X(F)$ be the smallest $p$ so that $F$ is a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for every choice of $A_i$ as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$. Alternatively, assuming $F$*

*is not a configuration in at least one of $I$, $I^c$, $T$, then $X(F)-1$ is the largest choice of $p$ so that $F$ is not a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for some choice of $A_i$ as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$.*

We are assuming $m$ is large and divisible by $p$, in particular that $m \geq (k+1)(k\ell+1)$ so that $m/p \geq k\ell+1$. Divisibility by $p$ does not affect the asymptotics since we can use a simple submatrix of a simple matrix that avoids $F$ for construction purposes. We are also using the fact that we need only consider $p$-fold products for $p \leq k+1$, since $F$ is a configuration in $\ell \cdot K_k$ and we can find $\ell \cdot K_k$ (and hence $F$) as a configuration in $A_1 \times A_2 \times \cdots \times A_{k+1}$ by taking 1 row from each of the first $k$ products (each row has [01]) and then, since we are taking no rows from the final $A_{k+1}$, we get the configuration $(m/(k+1)) \cdot K_k$ in the product. This also follows from Theorem 1.5.

If $F$ is a configuration in the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$, assume that $a_i$ rows of $A_i$ are used with $\sum_{i=1}^{p} a_i = k$. If we form the submatrix of $A_i$ of $a_i$ rows, then we would be interested in at most $\ell$ copies of a given column on these rows ($F$ has $\ell$ columns) if this is possible. Now for $t \geq k+\ell$, any $a_i$ rows of $K_t^1$ contains $\ell$ columns of 0's as well as a copy of $K_{a_i}^1$. The analogous result is true for $K_t^{t-1}$. Also for $t \geq kl+l$, the $a_i$ rows of $T_t$ consisting of rows $\ell+1, 2\ell+1, 3\ell+1, \ldots, k\ell+1$ have $\ell$ columns of 0's and $\ell \cdot T_{a_i}$. Thus as long as $m \geq (k+1)(k\ell+1)$ we are able to use the matrices $A_i$ as if they were arbitrarily large.

**Conjecture 2.2** *[32]*

$$forb(m,F) = \Theta(m^{X(F)-1}).\blacksquare$$

Note that the definition of $X(F)$ ensures forb$(m,F)$ is $\Omega(m^{X(F)-1})$, via the product construction, although for $X(F)=1$ a little care must be taken. The earliest use of the product construction is in [24] and its non trivial application in Theorem 2.6[24] and Theorem 3.4[24] for cases with $k=2$ and $k=3$. The Conjecture 2.2 has been verifed for $k=2$ in Theorem 3.2, $k=3$ in Theorem 4.1, $l=2$ in Theorem 6.2, $k=4$ and $F$ simple in Theorem 5.1, and other cases. Moreover the Conjecture has motivated recent work such as in Conjecture 7.1.

It is important to note that the constant in front of the leading term $m^{X(F)-1}$ of forb$(m,F)$ is not predicted by the Conjecture and so the Conjecture is little help with exact bounds.

Computing $X(F)$ is non trivial.

**Problem 2.3** *Show that computing $X(F)$ is NP-hard.*

Perhaps the problem Partition into Cliques would be useful. We have yet to make a direct connection between our proofs of asymptotic bounds for forb$(m,F)$ with the derivation of $X(F)$. We think of this problem as a configuration version of the Erdős-Stone-Simonovits [35] Theorem for the maximum number of edges in a graph avoiding some specified subgraph $H$ where $\chi(H)$ is relevant.

Some consequences of the conjecture can be considered problems.

**Problem 2.4** *Let forb$(m, F)$ be $\Theta(m^p)$. Is it true that forb$(m, t\cdot F)$ is $O(m^{p+1})$? Let forb$(m, 2\cdot F')$ be $\Theta(m^q)$. Is it true that forb$(m, t\cdot F')$ is $\Theta(m^q)$ for any $t \geq 2$?*

Problem 5.4 is a specific instance of this problem.

# 3 $F$ is a $1\times\ell$ or $2\times\ell$ (0,1)-matrix

For completeness we consider $1\times\ell$ $F$ (Theorem 5.1 and Corollary 5.2 from [19]).

**Theorem 3.1** *Assume $F$ is a $1\times\ell$ (0,1)-matrix with $p$ 1's and with $p \geq \ell - p \geq 0$ and let $F'$ be the $1\times p$ (0,1)-matrix with $p$ 1's. Assume $m \geq p-1 \geq 1$. Then*

$$\mathit{forb}(m, F') = \mathit{forb}(m, F) = \lfloor \frac{pm}{2} \rfloor + 1. \blacksquare$$

For the case $F$ is $2\times\ell$, the asymptotic classification of forb$(m, F)$ is completed in [24]. We need some special matrices

$$F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad F_2(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,0\,1 \\ 0\,0\cdots0\,1\,1 \end{bmatrix} \qquad F_3(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t} \\ 0\,0\cdots0\,1\cdots1 \end{bmatrix}$$

**Theorem 3.2** *Let $F$ be a $2\times\ell$ (0,1)-matrix.*
*(Constant Cases) If $F = F_1$, then forb$(m, F) = \Theta(1)$.*
*(Linear Cases) If $F$ has at least one configuration from $K_2^0$, $K_2^1$, $K_2^2$, $[2\cdot F_1]$, and if $F$ is a configuration in $F_2(t)$, $F_3(t)$, $F_3(t)^c$ for some $t \geq 1$, then forb$(m, F) = \Theta(m)$.*
*(Quadratic Cases) If $F$ has at least one configuration from $2\cdot K_2^0$, $[K_2^0|2\cdot K_2^1|K_2^2]$, or $2\cdot K_2^2$ then forb$(m, F) = \Theta(m^2)$.*
*In addition, any $2\times\ell$ (0,1)-matrix $F$ will fall into one of the three Cases.*

**Proof:** The linear bound for forb$(m, F_2(t))$ is Theorem 2.2[24]. The linear bound for forb$(m, F_3(t))$ is Theorem 2.3[24]. The quadratic construction for $[K_2^0|2\cdot K_2^1|K_2^2]$ is Theorem 2.6[24]. The quadratic bound in general for 2-rowed forbidden configurations follows from Theorem 1.5. All the lower bounds follow from the constructions given in Conjecture 2.2 but were developed in [24]. For example a linear construction for $2\cdot F_1$ is $I_m$. $\blacksquare$

A large number of exact or nearly exact bounds are available for 2-rowed $F$.

**Table 1** forb$(m, F)$ from [27].

| configuration $F$ | forb$(m, F)$ | reference |
|---|---|---|
| $\overbrace{\begin{bmatrix}0\cdots0\\1\cdots1\end{bmatrix}}^{q}$ | $\left\lfloor\frac{(q+1)m}{2}\right\rfloor + 2,\ m$ large | Thm 3.6[15] |
| $\begin{bmatrix}0\\1\end{bmatrix}$ | $2$ | [19] |
| $\begin{bmatrix}0&0\\1&1\end{bmatrix}$ | $m+2$ | [19] |
| $\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}$ | $2m+2$ | [19] |
| $\begin{bmatrix}0&0&0&0\\1&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{5m}{2}\right\rfloor+2$ | [27] |
| $\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}$ | $\left\lfloor\frac{3m}{2}\right\rfloor+1$ | [24] |
| $\begin{bmatrix}1&0&0&0\\0&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{7m}{3}\right\rfloor+1$ | [19] |
| $\begin{bmatrix}1&0&0&0&0\\0&1&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{11m}{4}\right\rfloor+1$ | [27] |
| $\begin{bmatrix}1&0&0&0&0&0\\0&1&1&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{15m}{4}\right\rfloor+1$ | [27] |
| $\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{8m}{3}\right\rfloor$ | [19] |
| $\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\end{bmatrix}$ | $\left\lfloor\frac{10m}{3}-\frac{4}{3}\right\rfloor$ | [27] |
| $\begin{bmatrix}1&1&0&0&0&0&0\\0&0&1&1&1&1&1\end{bmatrix}$ | $4m$ | [27] |
| $\begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{p}\\0\cdots0\,1\cdots1\end{bmatrix}$ | $pm-p+2$ | [19] |

An interesting case for which we do not know the exact bound is the following.

**Theorem 3.3** *[27], [19]*

$$\left(\frac{p+q}{2}+O(1)\right)m \le forb\left(m, \begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{q}\\0\cdots0\,1\cdots1\end{bmatrix}\right) \le qm-q+2. \blacksquare$$

From Theorem 2.6 and Corollary 2.7 of [19] we obtain:

**Theorem 3.4**

$$forb\left(m, \begin{bmatrix}0&1&1&1\\0&0&0&0\end{bmatrix}\right) = forb\left(m, \begin{bmatrix}0&1&1&1\\1&0&0&0\end{bmatrix}\right) = forb\left(m, \begin{bmatrix}0&1&1&1&0&1\\0&0&0&0&1&1\end{bmatrix}\right) = \left\lfloor\frac{7m}{3}\right\rfloor+1$$

From Theorem 2.3 and Corollary 2.5 of [19] we obtain:

**Theorem 3.5**

$$forb(m, \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}) = forb(m, \begin{bmatrix}0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}) = \lfloor \frac{3m}{2} \rfloor + 1. \blacksquare$$

We have the following exact bound (for large $m$) which is Theorem 1.3 in [15]. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount and for small $m$ the larger pigeonhole bound can be achieved.

**Theorem 3.6** *[15] Let $q \geq 3$ be given. Then for $m \geq \max\{5q-4, 8q-18\}$,*

$$forb(m, F = q \cdot \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\overbrace{\begin{matrix}1 & 1 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{matrix}}^{q}\end{bmatrix}) = \lfloor \frac{q+1}{2} m \rfloor + 2. \qquad \blacksquare \qquad (1)$$

Here is a table of bounds for 2-columned $F$ with 1 or 2 rows.

| configuration | forb$(m, F)$ | proof |
|---|---|---|
| $F_{1000} = \begin{bmatrix}1 & 1\end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F_{0100} = \begin{bmatrix}1 & 0\end{bmatrix}$ | $\binom{m}{0}$ | Thm 1.3 |
| $F_{2000} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ | $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F_{1100} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F_{1001} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0} + \binom{m}{m}$ | [19] or Thm 6.3 |
| $F_{0200} = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F_{0110} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |

**Theorem 3.7** *[19]*

$$forb(m, \begin{bmatrix}0\ \overbrace{11\cdots 1}^{p}\ 0\ 1\\ 0\ 00\cdots 0\ 1\ 1\end{bmatrix}) \leq (p - \frac{1}{2})m + 1. \blacksquare$$

Let us use the notation

$$F(r,p,q,s) = \begin{bmatrix}\overbrace{00\cdots 0}^{r}\ \overbrace{11\cdots 1}^{p}\ \overbrace{00\cdots 0}^{q}\ \overbrace{11\cdots 1}^{s}\\ 00\cdots 0\ 00\cdots 0\ 11\cdots 1\ 11\cdots 1\end{bmatrix}$$

**Theorem 3.8** *[19]* $forb(m, F(1,2,2,1)) = \lfloor \frac{m^2}{4} \rfloor + m + 1.$ ∎

**Theorem 3.9** *Assume $p \geq 4$. There exists an $m_0$ and a $c$ so that for $m \geq m_0$ and $m \geq 4(p-1)^{3/2}$,*

$$\frac{m^2}{4} + (p - 1\frac{1}{2} - \sqrt{p} - 1)m + O(p) \leq forb(m, F(1,p,p,1)) \leq \frac{m^2}{4} + (p-1)(m-2) + c. \blacksquare$$

**Theorem 3.10** *Assume $r \geq 2$, $r \geq p, q, s$. Then*

$$forb(m, F(r,0,0,0)) = forb(m, F(r,p,q,s)) = \frac{r+1}{6}m^2 + O(m)$$

The bounds do grow for larger $p$ as the coefficient of $m^2$ increases from $\frac{r+1}{6}$ to $\frac{r-1}{2}$.

**Theorem 3.11** *Assume $r, p, s \geq 2$ and $r \geq s$. Then*

$$forb(m, F(r,p,p,s)) \leq \frac{r-1}{2}m^2 + O(m)$$

*and for $r, s \geq 3$,*

$$\lim_{p \longrightarrow \infty} \frac{forb(m, F(r,p,p,s))}{m^2} = \frac{r-1}{2}$$

The following (Theorem 3.5 [19]) would be a useful (and somewhat surprising) tool in extending exact bounds.

**Theorem 3.12** *Assume $2 \leq p < q$. If there exist $a, b, c$ with $forb(m, F(r,p,p,s)) \leq am^2 + bm + c$ and $a, b > 0$, then there exists an $m_0$ (depending on $r, p, q, s, a$) so that for $m \geq m_0$ then $forb(m, F(r,p,q,s)) \leq am^2 + bm + c$*

# 4 $F$ is a $3 \times \ell$ (0,1)-matrix

For the case $F$ is $3 \times \ell$, the asymptotic classification of forb$(m, F)$ is begun in [24],[19] and was completed in [32]. The following configurations are needed for Theorem 4.1

$$F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_4(t) = \begin{bmatrix} 0\,\overbrace{1\cdots1}^{t}\,\overbrace{0\cdots0}^{t}\,0\,\overbrace{1\cdots1}^{t}\,1\,\overbrace{0\cdots0}^{t}\,1 \\ 0\,0\cdots0\,1\cdots1\,0\,1\cdots1\,0\,1\cdots1\,1 \\ 0\,0\cdots0\,0\cdots0\,1\,0\cdots0\,1\,1\cdots1\,1 \end{bmatrix}$$

$$F_5(t) = \begin{bmatrix} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 0 & 1 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 1 \\ 0 & 0\cdots0 & 1\cdots1 & 0 & 1 & 0\cdots0 & 1\cdots1 & 1 \\ 0 & 0\cdots0 & 0\cdots0 & 1 & 0 & 1\cdots1 & 1\cdots1 & 1 \end{bmatrix}$$

$$F_6(t) = \begin{bmatrix} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{1\cdots1}^{t} & \overbrace{1\cdots1}^{t} \\ 0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 & 0\cdots0 \\ 0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 \end{bmatrix}$$

**Theorem 4.1** *Let $F$ be a $3 \times \ell$ (0,1)-matrix.*
*(Linear Cases) If $F$ has at least one column and if $F$ is a configuration in $F_2$ then forb$(m, F) = \Theta(m)$.*
*(Quadratic Cases) If $F$ has at least one configuration from $K_3^0$, $K_3^1$, $K_3^2$, $K_3^3$, $2\cdot F_1$, $2\cdot F_1^c$ or $F_3$ and if $F$ is a configuration in $F_4(t)$, $F_5(t)$, $F_6(t)$ or $F_6(t)^c$ for some $t \geq 1$, then forb$(m, F) = \Theta(m^2)$.*
*(Cubic Cases) If $F$ has at least one configuration from $2 \cdot K_3^0$, $[2 \cdot K_3^1|K_3^2]$, $[2 \cdot K_3^1|K_3^3]$, $[K_3^0|2 \cdot K_3^2]$, $[K_3^1|2 \cdot K_3^2]$ or $2 \cdot K_3^3$ then forb$(m, F) = \Theta(m^3)$.*
*In addition, any $3 \times \ell$ (0,1)-matrix $F$ will fall into one of the three Cases.*

**Proof:** The linear bound for forb$(m, F_2)$ is Theorem 3.3[24]. The quadratic bound for forb$(m, F_4(t))$ is Theorem 3.9[24]. The quadratic bound for forb$(m, F_5(t))$ is Theorem 4.2 in [32] and the quadratic bound for forb$(m, F_6(t))$ is Theorem 4.1 in [32]. The cubic bound for all 3-rowed $F$ follows from Theorem 1.5 above. All the lower bounds follow from the constructions given in Conjecture 2.2 but had been developed as follows. Quadratic lower bounds for forb$(m, K_3^1)$, forb$(m, K_3^2)$, forb$(m, F_3)$ are in Corollary 3.5[24], quadratic lower bound for forb$(m, K_3^3)$ (and hence forb$(m, K_3^0)$ by taking the 0-1-complement) is in Theorem 3.6[24], quadratic lower bound for forb$(m, 2\cdot F_1)$ (and hence forb$(m, 2\cdot F_1^c)$) is in Theorem 3.7[24]. A cubic lower bound for forb$(m, 2\cdot K_3^3)$ (and hence forb$(m, 2 \cdot K_3^0)$) is in Theorem 3.9[24] and cubic lower bounds for forb$(m, [2 \cdot K_3^2|K_3^0])$ and forb$(m, [2 \cdot K_3^2|K_3^1])$ (and hence also for forb$(m, [2 \cdot K_3^1|K_3^3])$,forb$(m, [2 \cdot K_3^1|K_3^2])$) are in Theorem 3.10[24]. ■

There are a number of exact results.

**Theorem 4.2** *(Theorem 3.3 [24]) forb$(m, F_2) = 2m$* ■

**Theorem 4.3** *forb$(m, F_3) = \lfloor m^2/4 \rfloor + m + 1$* ■

**Proof:** The construction of taking $[K_{m/2}^0\,|\,T_{m/2}] \times [K_{m/2}^0\,|\,T_{m/2}]$ is Theorem 3.4 [24]. To prove the bound, one can use the shifting from Section 10 and Theorem 10.1. The number of different columns of $A_{|S}$ on a given set $S$ with $|S| = 3$ is at most 6 and so the same is true for the shifted matrix $T(A)$. But then since $T(A)$ is a downset, all columns in $T(A)$ have at most 2 1's and considering the columns of 2 1's as edges of a graph on a vertex set identified with the rows, we see that the graph has no triangles on any triple $S$ (or $T(A)_{|S}$ would have 7 different columns). And so by Mantel's bound (Turán) there are at most $\lfloor m^2/4 \rfloor$ columns of 2 1's and up to $m+1$ additional columns of less than 2 1's. ■

**Theorem 4.4** *[26]*

$$\mathit{forb}(m, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}) = \mathit{forb}(3 \cdot \mathbf{1}_2\mathbf{0}_0) = \frac{4}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$$

We have the following exact bound (for large $m$) which is Theorem 1.5 in [15]. Let $\mathbf{1}_k\mathbf{0}_l$ denote the $(k+l) \times 1$ vector of $k$ 1's on top of $l$ 0's. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount and for small $m$ the larger pigeonhole bound can be achieved.

**Theorem 4.5** *[15] Let $q > 2$ be given. There exists a constant $M$ so that for $m > M$,*

$$\mathit{forb}(m, q \cdot (\mathbf{1}_2\mathbf{0}_1) = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}}^{q} \\ \begin{matrix} 1 & 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}) \leq m + 2 + \frac{q+1}{3}\binom{m}{2} \tag{2}$$

*with equality for $m \equiv 1, 3 (mod\ 6)$.* ■

$3 \times 2$ **Forbidden Configurations**

| configuration | forb$(m, F)$ | proof |
|---|---|---|
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ | $\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ | Thm 1.7 |
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ | $\binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}$ | Thm 6.3 |
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $2m$ | Thm 3.3 in [24] |
| $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ | $\binom{m}{1} + \binom{m}{0} + \binom{m}{m}$ | Thm 3.2 in [24] (Thm 6.3) |
| $F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ | Thm 1.3 |
| $F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\lfloor 3m/2 \rfloor + 1$ | Thm 3.1 in [24] |

$3\times 3$ **Forbidden Configurations**

| configuration $F$ | forb$(m,F)$ | proof |
|---|---|---|
| $\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&0\\1&1&0\\1&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&0&0\\1&0&0\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.6 |
| $\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&0&1\\0&1&0\end{bmatrix}$ | $2m$ | [24] |
| $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ | $\frac{5}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.8 |
| $\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\1&1&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.7 |
| $\begin{bmatrix}1&1&1\\1&1&1\\0&0&0\end{bmatrix}$ | $\frac{4}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Thm 4.5 |
| $\begin{bmatrix}1&1&1\\1&1&1\\1&0&0\end{bmatrix}$ | $\frac{4}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 4.4 |
| $\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1\\1&1&0\\0&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Thm 4.6 |

It is an exercise to verify that all $3\times 3$ forbidden configurations (or their (0,1)-complements have been included in the table. We cannot complete the table for $3\times 4$ matrices but perhaps it is instructive to see how many are solved by the general results. I've organized the cases by first considering the number of columns of 3 1's and then the number of columns $\mathbf{1}^2\mathbf{0}_1$.

$3 \times 4$ **Forbidden Configurations**

| configuration | forb$(m, F)$ | proof |
|---|---|---|
| $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$ | $\frac{6}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.8 for $t=4$ |
| $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\end{bmatrix}$ | $\frac{5}{4}\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.8 for $t=3$ |
| $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.7 |
| $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&0&0&1\end{bmatrix}$, or $\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&0&0&0\end{bmatrix}$ | Exact bounds not known | |

| configuration | forb$(m,F)$ | proof |
| --- | --- | --- |
| $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&0&1&0\end{bmatrix}$, or $\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&0&0\end{bmatrix}$ | $\frac{5}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Theorem 4.5 |
| $\begin{bmatrix}1&1&1&1\\1&1&1&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&1\\1&1&1&0\\0&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&0\end{bmatrix}$ | Exact bounds not known | |
| $\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&1&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&0&1\end{bmatrix}$, $\begin{bmatrix}1&1&1&0\\1&1&0&0\\0&0&0&0\end{bmatrix}$ or $\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Thm 4.6 |

| configuration | forb$(m, F)$ | proof |
|---|---|---|
| $\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}, \begin{bmatrix}1&1&1&0\\1&1&0&1\\0&0&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\end{bmatrix}, \begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&0&0\end{bmatrix}$ | Exact bounds not known | |
| $\begin{bmatrix}1&1&0&1\\1&0&1&0\\0&1&1&0\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.6 |
| $\begin{bmatrix}1&1&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix},$ | 2m | Thm 4.2 |
| $\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\end{bmatrix}$ | $\left\lfloor\frac{m^2}{4}\right\rfloor+\binom{m}{1}+\binom{m}{0}$ | Thm 4.3 |

The following gives a number of exact results for 3-rowed $F$

**Theorem 4.6** *[26] Let $F$ be one of the following three matrices.*

$$\begin{bmatrix}1&1&1&0&0\\1&1&0&1&0\\0&0&1&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0&0\\1&1&0&1&0\\0&0&0&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&1&1&0&0\\1&1&0&0&0&0\\0&0&1&0&1&0\end{bmatrix}.$$

*Then for $m \geq 3$, forb$(m, F) =$ forb$(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) =$ forb$(m, \mathbf{1}_3\mathbf{0}_1)$.* ■

# 5 $F$ is a $4 \times \ell$ (0,1)-matrix

In this section we begin by considering $F$ to be itself a simple matrix. For the case that $F$ is simple and $4 \times \ell$, the asymptotic classification of forb$(m, F)$ was completed by Balin Fleming [20]. The main tools are Theorem 1.9 and Corollary 1.4.

We are able to establish the complete classification for the asymptotics of forb$(m, F)$ for any $4 \times \ell$ simple matrix $F$ and the result is consistent with the conjecture. To state the result we need a number of matrices.

$$F_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad F_2 = \begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix}, \quad F_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \quad F_4 = \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}, \quad F_5 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{bmatrix}$$

$$F_6 = \begin{bmatrix}1&0&1&1&1&0&1&1\\0&1&1&0&1&1&0&0\\0&0&1&1&0&1&0&1\\0&0&0&1&0&0&1&0\end{bmatrix}, F_7 = \begin{bmatrix}1&0&1&1&1&1&0&1\\0&1&1&1&1&0&1&0\\0&0&1&0&0&0&1&1\\0&0&0&1&0&1&0&0\end{bmatrix}, F_8 = \begin{bmatrix}1&0&1&0&0&1\\0&1&0&1&1&0\\0&0&1&1&1&1\\0&0&1&1&0&0\end{bmatrix}$$

$$F_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, F_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{11} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, F_{12} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, F_{13} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Theorem 5.1** *Let $F$ be a $4 \times \ell$ simple matrix.*
*(Linear Cases) If $F$ has a configuration $F_1$ and if $F$ is a configuration in $F_2$ then $forb(m, F) = \Theta(m)$.*
*(Quadratic Cases) If $F$ has a configuration $F_3$, $F_3^c$, $F_4$, $F_5$, or $F_5^c$ and if $F$ is a configuration in $F_6$, $F_7$, or $F_8$, then $forb(m, F) = \Theta(m^2)$.*
*(Cubic Cases) If $F$ has a configuration $K_4^0$, $F_9$, $F_9^c$, $F_{10}$, $F_{10}^c$, $F_{11}$, $F_{12}$, $F_{12}^c$, $F_{13}$, or $K_4^4$ then $forb(m, F) = \Theta(m^3)$.*
*In addition, any $4 \times \ell$ simple matrix $F$ will fall into one of the three Cases.*

**Proof:** The lower bounds are established by constructions in line with the conjecture. For definiteness, note that $F_1 \notin I$, $F_3 \notin I \times I$, $F_4 \notin T \times I$, $F_5 \notin I^c \times I^c$, $K_4^0, F_9, F_{10} \notin I^c \times I^c \times I^c$, $F_{11}, F_{12}, F_{13} \notin T \times T \times T$. The arguments are not entirely trivial. We see that any two rows of $I^c$ do not have $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and so a $k$-rowed matrix which has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ on every pair of rows is not a configuration in the $k-1$-fold product $I^c \times I^c \times \cdots \times I^c$. Similarly, any two rows of $T$ do not have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and so a $k$-rowed matrix which has $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on every pair of rows is not a configuration in the $(k-1)$-fold product $T \times T \times \cdots \times T$. This was noted following Corollary 3.5 in [24].

Theorem 6.2, Theorem 1.9, Theorem 1.3 establishes the upper bounds.

To show that any $4 \times \ell$ matrix $F$ is included in one of the three categories, assume that $F$ is a matrix that falls into neither the linear case or the cubic case. For convenience, think of a column of column sum 2 as an edge $(i, j)$ if the column has 1's in rows $i, j$. A matrix $F$ falls into the linear case only if $F = F_1$ or $F = F_2$. Examining the configurations $K_4^0$, $F_9$, $F_9^c$, $F_{10}$, $F_{10}^c$, $F_{11}$, $F_{11}$, $F_{12}$, $F_{12}^c$, $F_{13}$, $F_{13}^c$ or $K_4^4$, we deduce that $F$ cannot have a column of all 0's ($K_4^0$)or a column of all 1's ($K_4^4$). $F$ has at most two columns of column sum 1 and at most two columns of column sum 3 (using $F_{10}, F_{10}^c$). In addition four edges forming a four cycle yields $F_{11}$ and so there are at most 4 edges in $F$ which must be a subgraph of a triangle plus one edge from the triangle to the remaining vertex. (From this and Corollary 1.4 it follows that any 4-rowed configuration with a quadratic bound has at most 8 column types).

If $F$ has no columns of either three 1's or three 0's then, assuming it is not $F_1$ or $F_2$, it must contain two disjoint edges and hence $F_4$ or have three columns of column sum 2 forming a triangle ($F_5^c$) or three columns of column sum 2 sharing a vertex ($F_5$). ■

For the general case $F$ is $4\times\ell$, the asymptotic classification of forb$(m,F)$ is not complete but we can use the conjecture to predict the answer. The following configurations are needed for Conjecture 5.2

$$F_6(t) = \left[\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix} t\cdot\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&1&0&1\\0&0&1&0\end{bmatrix}\right]$$

$$F_7(t) = \left[\begin{bmatrix}1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{bmatrix} t\cdot\begin{bmatrix}1&1&0&1\\1&0&1&0\\0&0&1&1\\0&1&0&0\end{bmatrix}\right]$$

$$F_8(t) = \left[\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&1&1\end{bmatrix} t\cdot\begin{bmatrix}0&1\\1&0\\1&1\\0&0\end{bmatrix}\right]$$

$$B_1 = \begin{bmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&1&1\\0&1&1&1&1\end{bmatrix}, B_2 = \begin{bmatrix}0&0&0&1&1\\0&0&0&1&1\\0&0&1&1&1\\0&1&1&0&1\end{bmatrix}, B_3 = \begin{bmatrix}0&0&0&1&1\\0&0&0&1&1\\0&0&1&0&1\\0&1&1&1&1\end{bmatrix},$$

$$B_4 = \begin{bmatrix}0&0&0&0&1\\0&0&1&1&1\\0&0&1&1&1\\0&1&0&1&1\end{bmatrix}, B_5 = \begin{bmatrix}0&0&0&1&1\\0&0&1&1&1\\0&0&1&0&1\\0&1&0&1&1\end{bmatrix}, B_6 = \begin{bmatrix}0&0&0&1&1\\0&0&1&1&1\\0&0&1&1&1\\0&1&0&0&1\end{bmatrix}.$$

$$D_{12} = \begin{bmatrix}0&0&0&0&0&0&0&1&1&1&1\\0&0&0&1&1&1&1&0&0&0&0\\1&0&1&0&1&0&1&0&1&0&1\\0&1&1&0&0&1&1&0&0&1&1\end{bmatrix}.$$

**Conjecture 5.2** *Let $F$ be a $4\times\ell$ (0,1)-matrix.*
*(Linear Cases) If $F$ has $F_1$ as a configuration and if $F$ is a configuration in $F_2$ then forb$(m,F)=\Theta(m)$.*
*(Quadratic Cases) If $F$ has at least one configuration from $F_3$, $F_3^c$, $F_4$, $F_5$, $F_5^c$ or $2\cdot F_1$, and if $F$ is a configuration in $F_6(t)$, $F_7(t)$ or $F_8(t)$ for some $t$ , then forb$(m,F)=\Theta(m^2)$.*
*(Cubic Cases) If $F$ has at least one configuration from $K_4^0$, $2\cdot F_3$, $F_9$, $F_9^c$, $F_{10}$, $F_{10}^c$, $F_{11}$, $F_{12}$, $F_{12}^c$, $F_{13}$, $2\cdot F_3^c$ or $K_4^4$ and if $F$ is a configuration in $[K_4\,|\,t\cdot[K_4-B_i]]$ or $[K_4\,|\,t\cdot[K_4-B_i]]^c$ for $i=1,2,\ldots 6$ or $[K_4^0\,|\,t\cdot[K_4^1|D_{12}]]$ then forb$(m,F)=\Theta(m^3)$.*
*(Quartic Cases) If $F$ has at least one configuration from $2\cdot K_4^0$, $[2\cdot K_4^2]$, $[2\cdot K_4^4]$ or $[2\cdot K_4^1|C]$ or $[2\cdot K_4^1|C]^c$ where $C$ is one of $K_4^2$, $F_{12}$, $F_9^c$, $F_{10}^c$, $K_4^4$, then forb$(m,F)=\Theta(m^4)$.*
*In addition, any $4\times\ell$ (0,1)-matrix $F$ will fall into one of the four cases.*

The boundary between linear and quadratic follows easily from Theorems 6.1,6.2. The boundary between quadratic and cubic is mostly not proven and examples are given below. The cubic lower bounds are from the constructions. The boundary between cubic and quartic is in Theorem 1.10 and Theorem 1.11. Obviously we need the quartic bound of Theorem 1.5.

We have already proved (one can use induction or the results in [22]):

**Theorem 5.3** *[14] forb*$(m, t\cdot F_2)$ *is* $\Theta(m^2)$. ■

The Conjecture 2.2 predicts the following and it would be a helpful first step.

**Problem 5.4** *Is forb*$(m, t\cdot F_4)$ *equal to* $\Theta(m^2)$ *for* $t \geq 3$*?* ■

An argument special for the case $t = 2$ proves the following:

**Theorem 5.5** *[14] forb*$(m, 2\cdot F_4)$ *is* $\Theta(m^2)$.

We have an exact bound for $F_4 = F_{0,2,2,0}$. This can be viewed as a variation of a result of Kleitman [45]. In that result the condition was that pairs of sets $B, C$ have $|B\backslash C|+|C\backslash B| \leq 2t$. The condition of forbidding $F_4$ is slightly weaker than the condition for $t = 1$ and so the bound for the result below is slightly larger than Kleitman's bound. This is modest progress for Problem 13.2.

**Theorem 5.6** *forb*$(m, F_4) = \binom{m}{2} + m - 2$. ■

We have the following exact bound (for large $m$) which is Theorem 1.6 in [15]. A pigeonhole argument yields a bound that exceeds the bound below by a linear amount and for small $m$ the larger pigeonhole bound can be achieved.

**Theorem 5.7** *Let* $q > 2$ *be given. There exists a constant* $M$ *so that for* $m > M$,

$$forb(m, q\cdot F_1) = \Bigg( \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q} \Bigg) \leq 2 + 2m + \frac{q+3}{3}\binom{m}{2}, \tag{3}$$

*with equality if in addition* $m \equiv 1, 3 (mod\, 6)$. *Moreover, for* $m > M$, *if the bound is achieved by an* $m$*-rowed matrix* $A$, *then* $A$ *has all columns of sum 0, 1, 2,* $m-2$, $m-1$, $m$ *and for some integers* $a, b$ *with* $a + b = q - 3$, *the columns of sum 3 correspond to a simple* $(m, 3, a)$*-design and the columns of sum* $m-3$ *correspond to the (0,1)-complement of a simple* $(m, 3, b)$*-design and there are no other columns.* ■

$4 \times 2$ **Forbidden Configurations**

| configuration | forb$(m,F)$ | proof |
|---|---|---|
| $F_{4000} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ | $\binom{m}{4}+\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $F_{3100} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $F_{3001} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Thm 6.3 |
| $F_{2200} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\geq (\frac{29}{21})\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ <br> $\leq 2\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | [16] |
| $F_{2,1,0,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | Thm 6.3 |
| $F_{2,0,0,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m-1}+\binom{m}{m}$ | Thm 6.3 |
| $F_{1,3,0,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |

| configuration | forb$(m,F)$ | proof |
|---|---|---|
| $F_{1,2,1,0} = \begin{bmatrix}1&1\\1&0\\1&0\\0&1\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | [16] |
| $F_{1,2,0,1} = \begin{bmatrix}1&1\\1&0\\1&0\\0&0\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | [16] |
| $F_{1,1,1,1} = \begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix}$ | $4m-4$ | [16] |
| $F_{0,4,0,0} = \begin{bmatrix}1&0\\1&0\\1&0\\1&0\end{bmatrix}$ | $\binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ | Thm 1.3 |
| $F_{0,3,1,0} = \begin{bmatrix}1&0\\1&0\\1&0\\0&1\end{bmatrix}$ | $\binom{m}{2}+\binom{m}{1}+\binom{m}{0}+\binom{m}{m}$ | [16] |
| $F_{0,2,2,0} = \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}$ | $\binom{m}{2}+2m-1$ | Thm 5.6 |

The following suggests that an exact bound for $F_{2,1,1,0}$ would be difficult. In a similar way one expects that there an exact bound for $F_{a,1,1,0}$ would be difficult.

**Theorem 5.8** *Let $c$ be a positive real number. Let $A$ be an $m \times \left(c\binom{m}{2}+m+2\right)$ simple matrix with no $F_{2,1,1,0}$. Then for some $M > m$, there is an $M \times \left((c+\frac{2}{m(m-1)})\binom{M}{2}+M+2\right)$ simple matrix with no $F_{2,1,1,0}$.* ■

The following are in [43].

**Theorem 5.9** *Let $F = \begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}$. Then for $m \geq 3$ we have forb$(m,F) = \binom{m}{2}+m+2$.* ■

**Theorem 5.10** *Let $F$ be one of the following three matrices.*

$$\begin{bmatrix}1&1&1&1&1&0&0&0&0&0\\1&1&1&0&0&1&1&1&0&0\\1&1&0&1&0&1&1&0&1&0\\0&0&1&0&0&0&0&1&0&0\end{bmatrix} \quad or \quad \begin{bmatrix}1&1&1&1&1&0&0&0&0&0\\1&1&1&0&0&1&1&1&0&0\\1&1&0&1&0&1&1&0&1&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$or \quad \begin{bmatrix}1&1&1&1&1&1&0&0&0&0&0&0\\1&1&1&1&0&0&1&1&1&1&0&0\\1&1&0&0&0&0&1&1&0&0&0&0\\0&0&1&0&1&0&0&0&1&0&1&0\end{bmatrix}.$$

*Then for $m \geq 4$, $forb(m, F) = forb(m, 2 \cdot \mathbf{1}_3\mathbf{0}_1) = forb(m, \mathbf{1}_4\mathbf{0}_1) = \binom{m}{3} + \binom{m}{2} + m + 2$.*■

# 6 $F$ is a $k \times 1$ or $k \times 2$ (0,1)-matrix.

For completeness, we first consider $k \times 1$ $F$. Let $\mathbf{1}_a\mathbf{0}_b$ denote the column of $a$ 1's on top of $b$ 0's.

**Theorem 6.1** *Let $s, k$ be given positive integers with $s \leq k$. Then*

$$forb(m, \mathbf{1}_s\mathbf{0}_{k-s}) = \sum_{i=0}^{s-1} \binom{m}{i} + \sum_{i=0}^{k-s-1} \binom{m}{i}. \blacksquare$$

For the case $F$ is $k \times 2$, the asymptotic classification of forb$(m, F)$ is in [25]. Let $F_{a,b,c,d}$ be the $(a + b + c + d) \times 2$ (0,1)-matrix which has $a$ rows of [11], $b$ rows of [10], $c$ rows of [01], $d$ rows of [00]. By interchanging columns we see that forb$(m, F_{a,b,c,d}) =$ forb$(m, F_{a,c,b,d})$, and by considering (0,1)-complements we see that forb$(m, F_{a,b,c,d}) =$ forb$(m, F_{d,c,b,a})$. Therefore we may assume that $a \geq d$ and $b \geq c$. Our result for the function forb$(m, F_{a,b,c,d})$ is the following.

**Theorem 6.2** *[25] Suppose $a \geq d$ and $b \geq c$. Then $forb(m, F_{a,b,c,d})$ is $\Theta(m^{a+b-1})$ if either $b > c$ or $a, b \geq 1$. Also $forb(m, F_{a,0,0,d})$ is $\Theta(m^a)$ and $forb(m, F_{0,b,b,0})$ is $\Theta(m^b)$.*

Our main technique is a strong stability result Theorem 13.1 and induction such as Lemma 9.2. We have obtained a number of exact results.

**Theorem 6.3** *[16]Assume $a, d, m$ are given integers with $a \geq d$ and $m \geq a + d$, then*

$$forb(m, 2 \cdot \mathbf{1}_a\mathbf{0}_d) = forb(m, F_{a,0,0,d}) = forb(m, F_{a,1,0,d}) = \sum_{j=0}^{a} \binom{m}{j} + \sum_{j=m-d+1}^{m} \binom{m}{j}. \blacksquare$$

**Theorem 6.4** *[26] Let $m, a, b$ be given integers. For $m \geq 1$, $a \geq 2$ and $b \geq 2$,*

$$forb(m, F_{a,b,0,1}) = forb(m, F_{a,b,1,0}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_1)$$

$$\text{and } forb(m, F_{a,b,1,1}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_2).$$

*Also for $a \geq 2$,*

$$forb(m, F_{a,1,0,1}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_1),$$

*and for $b \geq 2$,*

$$forb(m, F_{1,b,1,0}) = forb(m, \mathbf{1}_{1+b}\mathbf{0}_1),$$

$$forb(m, F_{1,b,1,1}) = forb(m, \mathbf{1}_{1+b}\mathbf{0}_2).$$

*Also for $b \geq 3$ [16],*

$$forb(m, F_{0,b,1,0}) = forb(m, \mathbf{1}_b\mathbf{0}_1)\blacksquare$$

**Problem 6.5** *Assume we are given positive integers $a, b, c, d$ with $a \geq d$ and $b \geq c$. Find some mild conditions on $a, b, c, d$ so that $forb(m, F_{a,b,c,d}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$.* $\blacksquare$

# 7 $F$ is a simple $5 \times \ell$ matrix

For the case that $F$ is a $5 \times \ell$ simple matrix, We can use Conjecture2.2 to predict the results. The non trivial calculations to achieve this are in [29]. Some of the asymptotics have proofs. The numbered matrices are given after the conjecture. Theorem 1.9 establishes the cubic bounds. The quadratic bounds for $F_3, F_4, \ldots F_{11}$ are an open problem.

**Conjecture 7.1** *Let $F$ be a $5 \times \ell$ simple matrix.*
*(Quadratic Cases) If $F$ has a configuration of $F_1$ or $F_2$ and if $F$ is a configuration in $F_3, F_4, \ldots, F_{11}$ then $forb(m, F)$ is $\Theta(m^2)$.*
*(Cubic Cases) If $F$ has a configuration of one of $F_{12}, F_{13}, \ldots, F_{24}$ and if $F$ is a configuration in $F_{25}, F_{26}, \ldots, F_{29}$ then $forb(m, F)$ is $\Theta(m^3)$.*
*(Quartic Cases) If $F$ has a configuration one of $F_{30}, F_{31}, \ldots, F_{87}$ then $forb(m, F)$ is $\Theta(m^4)$.*
*In addition, any $5 \times \ell$ simple matrix $F$ will fall into one of these three cases.*

.

$$F_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

**Maximal quadratics** (by conjecture)

$$F_3 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$F_6 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad F_8 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_9 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad F_{10} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{11} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

**Minimal Cubics** (by conjecture)

$$F_{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad F_{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad F_{14} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad F_{15} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_{16} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad F_{17} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad F_{18} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$F_{19} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad F_{20} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad F_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$F_{22} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad F_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad F_{24} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Maximal Cubics by Theorem 1.9**

$$F_{25} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{26} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_{27} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{28} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{29} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

**Minimal quartics by Theorem 1.9**

$$F_{30} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad F_{31} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_{32} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad F_{33} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad F_{34} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$F_{35} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad F_{36} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad F_{37} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{38} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad F_{39} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad F_{40} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$F_{41} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad F_{42} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad F_{43} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{44} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad F_{45} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{46} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$F_{47} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad F_{48} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad F_{49} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$F_{50} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad F_{51} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{52} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{53} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad F_{54} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad F_{55} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$F_{56} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad F_{57} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad F_{58} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$F_{59} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{60} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad F_{61} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_{62} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{63} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad F_{64} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$F_{65} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad F_{66} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_{67} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_{68} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad F_{69} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad F_{70} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{71} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad F_{72} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad F_{73} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{74} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad F_{75} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad F_{76} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$F_{77} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&1&1&0&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \quad F_{78} = \begin{bmatrix} 1&0&0&0&1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1\\ 1&1&1&1&0 \end{bmatrix} \quad F_{79} = \begin{bmatrix} 0&1&1&1&0\\ 1&0&1&1&0\\ 1&1&0&1&0\\ 1&1&1&0&0\\ 0&0&0&0&1 \end{bmatrix}$$

$$F_{80} = \begin{bmatrix} 1&0&0&1&1\\ 0&1&0&1&1\\ 0&0&1&0&1\\ 0&0&1&1&0\\ 1&1&0&0&1 \end{bmatrix} \quad F_{81} = \begin{bmatrix} 0&1&1&0&0\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&1&0&0&1\\ 0&0&1&1&0 \end{bmatrix} \quad F_{82} = \begin{bmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&1&1&1\\ 0&1&1&0&1&0\\ 1&0&0&1&0&1 \end{bmatrix}$$

$$F_{83} = \begin{bmatrix} 0&1&1&1&0&0\\ 1&0&1&1&0&0\\ 1&1&0&0&0&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0 \end{bmatrix} \quad F_{84} = \begin{bmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&1&0&1\\ 0&1&1&0&0&1\\ 1&0&0&1&1&0 \end{bmatrix} \quad F_{85} = \begin{bmatrix} 0&1&1&1&0&0\\ 1&0&1&1&0&0\\ 1&1&0&0&1&0\\ 1&0&0&1&1&0\\ 0&1&1&0&0&1 \end{bmatrix}$$

$$F_{86} = \begin{bmatrix} 1&0&0&0&0&1\\ 0&1&1&0&0&1\\ 0&0&0&1&1&1\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \quad F_{87} = \begin{bmatrix} 0&1&1&1&1&0\\ 1&0&0&1&1&0\\ 1&1&1&0&0&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{bmatrix} \blacksquare$$

# 8 What is missing if a configuration $F$ is avoided?

Let $F$ be a given $k \times \ell$ (0,1)-matrix. Let $S$ be a subset of $[m]$, the rows of $A$. Let us use the notation $A_{|S}$ to denote the submmatrix of $A$ consisting of the rows indexed by $S$. We are interested in what conditions on $A_{|S}$ must be satisfied so that $A$ has no configuration $F$. The problem of forbidden configurations does not reduce to these conditions since the conditions do not refer to the simplicity of $A$ but these conditions have been used successfully.

We say an $|S| \times 1$ column $\alpha$ on a set of rows $S$ is in 'short supply' in $A$ if $A_{|S}$ has at most some constant number of columns equal to $\alpha$. In this circumstance row order is relevant. We are not considering columns of $A_{|S}$ up to row permutations.

A careful consideration is required to see what is missing from $A$ when a configuration $F$ is not a configuration in $A$. The following is an example from cases with $k = 3$. Let $\{i, j, k\}$ be a triple of rows of a matrix $A = (a_{rs})$. We say that we have

$$\text{no}\ \begin{matrix} i\\ j\\ k \end{matrix} \begin{bmatrix} d\\ e\\ f \end{bmatrix} \tag{4}$$

if in every column $q$ of $A$ we do not have $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occurring. As well, we say that there are

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{5}$$

if there are at most $t-1$ columns $q$ of $A$ in which $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occur.

Let $S_p$ denote the symmetric group on $p$ symbols.

**Proposition 8.1** *(Proposition 2.1[32]) Let A be a (0,1)-matrix with no configuration $F_6(t)$ of Section 4. Let $a, b, c$ be a triple of rows of $A$. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ (note that $\{a, b, c\}$ and $\{i, j, k\}$ are the same as sets) with*

$$\text{no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6}$$

*or if we do not have (6), then we have a permutation $\pi_2 \in S_3$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{7}$$

*or if we do not have (6),(7), then we have a permutation $\pi_3 \in S_3$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

**Proof:** (sample) If one of (6),(7),(8) is true we have no $F_6(t)$. Give (6) is false, we either have $t \cdot K_3^1$ in the triple of rows or not. If not, then (7) holds for some ordering. If we do have $t \cdot K_3^1$ in the triple of rows, then $t$ copies of two columns of two 1's (in the triple of rows) will yield $F_6(t)$ and so at most one column of two 1's appears $t$ or more times. Thus (8) holds. ■

**Proposition 8.2** *(Proposition 2.2[32]) Let A be a (0,1)- matrix with no configuration $F_5(t)$ of Section 4. Let $a, b, c$ be a triple of rows of $A$. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ with*

$$\text{no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

*or if we do not have (9), then we have a permutation $\pi_2 \in S_3$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{10}$$

*or if we do not have (9),(10), then we have a permutation $\pi_3 \in S_3$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{11}$$

*or if we do not have (9),(10),(11), then we have a permutation $\pi_4 \in S_3$ with $\pi_4(a) = i$, $\pi_4(b) = j$, $\pi_4(c) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \blacksquare \tag{12}$$

We can readily establish such results for various $F$ but it does take some careful thought. The results for the 4-rowed and 5-rowed $F = [K_k^0 \,|\, t \cdot [2 \cdot K_k^1\, D_{12}]$ from Problem 1.11 are given below and may be useful in solving Problem 1.11.

**Proposition 8.3** *Let $A$ be a (0,1)-matrix with no configuration 4-rowed configuration $F_6(t) = [K_4^0 \,|\, t \cdot [2 \cdot K_4^1\, D_{12}]$ from Theorem 1.11. Let $a, b, c, d$ be four of rows of $A$. Then we either have a permutation $\pi_1 \in S_4$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$, $\pi_1(d) = l$ (note that $\{a, b, c, d\}$ and $\{i, j, k, l\}$ are the same as sets) with*

$$\text{no } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{13}$$

*or if we do not have (13), then we have a permutation $\pi_2 \in S_4$ with $\pi_2(a) = i$, $\pi_2(b) = j$, $\pi_2(c) = k$, $\pi_2(d) = l$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

*or if we do not have (13),(14), then we have a permutation $\pi_3 \in S_4$ with $\pi_3(a) = i$, $\pi_3(b) = j$, $\pi_3(c) = k$, $\pi_3(d) = l$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{15}$$

*or if we do not have (13),(14),(15), then we have a permutation* $\pi_4 \in S_4$ *with* $\pi_4(a) = i$, $\pi_4(b) = j$, $\pi_4(c) = k$, $\pi_4(d) = l$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{16}$$

*or if we do not have (13),(14),(15),(16), then we have a permutation* $\pi_5 \in S_4$ *with* $\pi_5(a) = i$, $\pi_5(b) = j$, $\pi_5(c) = k$, $\pi_5(d) = l$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{17}$$

*or if we do not have (13),(14),(15),(16),(17), then we have a permutation* $\pi_6 \in S_4$ *with* $\pi_6(a) = i$, $\pi_6(b) = j$, $\pi_6(c) = k$, $\pi_6(d) = l$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \\ l \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \blacksquare \tag{18}$$

**Proposition 8.4** *[17] Let A be a (0,1)-matrix with no configuration 5-rowed configuration* $F_6(t) = [K_5^0 \,|\, t \cdot [2 \cdot K_5^1\, D_{12}]$ *from Theorem 1.11 where*

$$F_6(t) = \left[ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \;\; t \cdot \left[ \begin{matrix} \begin{bmatrix} 00 & \cdots & 0 & 11 & \cdots & 1 & 00 & \cdots & 0 \\ 00 & \cdots & 0 & 00 & \cdots & 0 & 11 & \cdots & 1 \end{bmatrix} \\ K_3 - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad K_3 \qquad K_3 \end{matrix} \right] \right]. \tag{19}$$

*Let a,b,c,d,e be five rows of A. Then we either have a permutation* $\pi_1 \in S_5$ *with* $\pi_1(a) = g$, $\pi_1(b) = h$, $\pi_1(c) = i$, $\pi_1(d) = j$, $\pi_1(e) = k$ *with*

$$\text{no } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{20}$$

*or if we do not have (20), then we have a permutation* $\pi_2 \in S_5$ *with* $\pi_2(a) = g$, $\pi_2(b) = h$, $\pi_2(c) = i$, $\pi_2(d) = j$, $\pi_2(e) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{21}$$

*or if we do not have (20),(21), then we have a permutation* $\pi_3 \in S_5$ *with* $\pi_3(a) = g$, $\pi_3(b) = h$, $\pi_3(c) = i$, $\pi_3(d) = j$, $\pi_3(e) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{and at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{22}$$

*or if we do not have (20),(21),(22), then we have a permutation* $\pi_4 \in S_5$ *with* $\pi_4(a) = g$, $\pi_4(b) = h$, $\pi_4(c) = i$, $\pi_4(d) = j$, $\pi_4(e) = k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{and at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

*or if we do not have (20)-(23), then we have a permutation* $\pi_5 \in S_5$ *with* $\pi_5(a) = g$, $\pi_5(b) = h$, $\pi_5(c) = i$, $\pi_5(d) = j$, $\pi_5(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{24}$$

*or if we do not have (20)-(24) then we have a permutation* $\pi_6 \in S_5$ *with* $\pi_6(a) = g$, $\pi_6(b) = h$, $\pi_6(c) = i$, $\pi_6(d) = j$, $\pi_6(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{25}$$

*or if we do not have (20)-(25) then we have a permutation* $\pi_7 \in S_5$ *with* $\pi_7(a) = g$, $\pi_7(b) = h$, $\pi_7(c) = i$, $\pi_7(d) = j$, $\pi_7(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{and } \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \tag{26}$$

*or if we do not have (20)-(26) then we have a permutation* $\pi_8 \in S_5$ *with* $\pi_8(a) = g$, $\pi_8(b) = h$, $\pi_8(c) = i$, $\pi_8(d) = j$, $\pi_8(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{27}$$

*or if we do not have (20)-(27) then we have a permutation* $\pi_9 \in S_5$ *with* $\pi_9(a) = g$, $\pi_9(b) = h$, $\pi_9(c) = i$, $\pi_9(d) = j$, $\pi_9(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{28}$$

*or if we do not have (20)-(28), then we have a permutation* $\pi_{10} \in S_5$ *with* $\pi 10(a) = g$, $\pi_{10}(b) = h$, $\pi_{10}(c) = i$, $\pi_{10}(d) = j$, $\pi_{10}(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

*or if we do not have (20)-(29) then we have a permutation* $\pi_{11} \in S_5$ *with* $\pi_{11}(a) = g$, $\pi_{11}(b) = h$, $\pi_{11}(c) = i$, $\pi_{11}(d) = j$, $\pi_{11}(e) = k$ *with*

$$\leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, and \ \leq t-1 \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{29}$$

*or if we do not have (20)-(29) then we have a permutation* $\pi_{12} \in S_5$ *with* $\pi_{12}(a) = g$, $\pi_{12}(b) = h$, $\pi_{12}(c) = i$, $\pi_{12}(d) = j$, $\pi_{12}(e) = k$ *with*

$$at \ most \ t-1 \ of \ \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, and \ at \ most \ t-1 \ of \ \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

*or if we do not have (20)-(30) then we have a permutation* $\pi_{13} \epsilon S_5$ *with* $\pi_{13}(a) = g$, $\pi_{13}(b) = h$, $\pi_{13}(c) = i$, $\pi_{13}(d) = j$, $\pi_{13}(e) = k$ *with*

$$at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, and\ at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{31}$$

*or if we do not have (20)-(31) then we have a permutation* $\pi_{14} \epsilon S_5$ *with* $\pi_{14}(a) = g$, $\pi_{14}(b) = h$, $\pi_{14}(c) = i$, $\pi_{14}(d) = j$, $\pi_{14}(e) = k$ *with*

$$at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, and\ at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{32}$$

*or if we do not have (20)-(32) then we have a permutation* $\pi_{15} \epsilon S_5$ *with* $\pi_{15}(a) = g$, $\pi_{15}(b) = h$, $\pi_{15}(c) = i$, $\pi_{15}(d) = j$, $\pi_{15}(e) = k$ *with*

$$at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, and\ at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, and\ at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{33}$$

*or if we do not have (20)-(33) then we have a permutation* $\pi_{16} \epsilon S_5$ *with* $\pi_{16}(a) = g$, $\pi_{16}(b) = h$, $\pi_{16}(c) = i$, $\pi_{16}(d) = j$, $\pi_{16}(e) = k$ *with*

$$at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, and\ at\ most\ t-1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{34}$$

*or if we do not have (20)-(34) then we have a permutation* $\pi_{17} \epsilon S_5$ *with* $\pi_{17}(a) = g$, $\pi_{17}(b) = h$, $\pi_{17}(c) = i$, $\pi_{17}(d) = j$, $\pi_{17}(e) = k$ *with*

$$at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, and\ at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, and\ at\ most\ t{-}1\ of\ \begin{array}{c} g \\ h \\ i \\ j \\ k \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{35}$$

*or if we do not have (20)-(35) then we have a permutation $\pi_{18} \in S_5$ with $\pi_{18}(a) = g$, $\pi_{18}(b) = h$, $\pi_{18}(c) = i$, $\pi_{18}(d) = j$, $\pi_{18}(e) = k$ with*

$$\text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{at most } t-1 \text{ of } \begin{matrix} g \\ h \\ i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \blacksquare \tag{36}$$

# 9 Standard Induction

There are easy standard inductions based on either deleting the first row or perhaps the first two rows. The most attractive application is the bound Theorem 1.9 but there are many other applications.

The *standard argument* [24] proceeds as follows. Let $A$ be a simple $m \times n$ matrix with no configuration $F$ or some such property. Then we can decompose $A$ as

$$A = \begin{bmatrix} 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\ B & C & C & D \end{bmatrix} \tag{37}$$

where $C$ is chosen as those columns which are repeated in the matrix obtained from $A$ by deleting the first row and then we have reordered the columns of $A$ to obtain the decomposition above. Thus $[BCD]$ is simple and configurations forbidden in $A$ are also forbidden in the $(m-1)$-rowed matrix $[BCD]$. Also the number of columns in $A$ is the number of columns in $[BCD]$ and the number of columns in $C$. One can easily derive the upper bound of Theorem 1.3 this way by noting that if $A$ has no $K_k$ then $C$ has no $K_{k-1}$. Thus $\text{forb}(m, K_k) \le \text{forb}(m-1, K_k) + \text{forb}(m, K_{k-1})$. A typical application of the standard argument is the Lemma below, a version is stated in [25].

**Lemma 9.1** *[26] Let $k$ be given and let $F$ be a $k$-rowed matrix. For each $s \in [k]$, decompose $F$ as*

$$F = \begin{bmatrix} 0\,0\cdots0 & 0\,0\cdots0 & 1\,1\cdots1 & 1\,1\cdots1 \\ B_s(F) & C_s(F) & C_s(F) & D_s(F) \end{bmatrix} \leftarrow \textit{row } s \tag{38}$$

*where we have permuted the rows of $F$ so row $s$ is the first row and $C_s(F)$ consists of the repeated columns after deleting that row from $F$. Then if $A$ is a simple matrix with no configuration $F$, then in the row decomposition of $A$ of (37), we deduce that $C$ has no configurations $F_s = [B_s(F)C_s(F)D_s(F)]$ for each $s \in [k]$. In particular if forb$(m, \{F_1, F_2, \ldots, F_k\})$ is $O(m^t)$ then forb$(m, F)$ is $O(m^{t+1})$.* $\blacksquare$

An application of the standard argument in [30] is concerned with the row order which is typically preserved by the induction (one could imagine other versions where we select the row to induct upon). Another useful induction is the following. Decompose the columns of $A$ into either the column of 0's or the column of 1's or the columns with a 0 in row 1 (but not the column of 0's) or the columns with a 1 in row 1 (but not the column of 1's). Then the columns with a 0 in row 1 which are not all 0's will have a first 1 and so, for each $i = 2, 3, \ldots, m$, we can consider the set of columns whose first 1 is in row $i$. Similarly we can consider the set of columns whose first 0 is in row $i$. for each $i = 2, 3, \ldots, m$. An application of this induction is in Theorem 6.2.

**Lemma 9.2** *Let $F$ be a $k \times \ell$ $(0,1)$-matrix for which forb$(m, F)$ is $O(m^t)$. Then with*

$$F' = \begin{bmatrix} 11\cdots1 \\ 00\cdots0 \\ F \end{bmatrix}$$

*we have forb$(m, F')$ being $O(m^{t+1})$.*

A two rowed induction was used with success in [31] in the case that the columns of matrix $A$ form an antichain as sets. Using that fact, we can deduce that $B_2$ is empty in the decomposition (37) above and so we may write

$$A = \begin{bmatrix} 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\ 00\cdots0 & 11\cdots1 & 00\cdots0 & 11\cdots1 \\ C_1 & C_2C_3 & C_3C_4 & C_5 \end{bmatrix}$$

where $[C_1C_2C_3C_4C_5]$ is simple.

# 10 Shifting proofs

Peter Frankl popularized the use of shifting arguments in extremal set theory. In this particular context there is a paper of Frankl [36] and a paper of Alon [3] using shifting techniques to generalize the Sauer Bound of Theorem 1.3. I extended these arguments and used them in [10]. Shifting is easily defined in set language. Let $\mathcal{F} \subseteq 2^{[m]}$. Let

$$T_j(B) = \begin{cases} B & \text{if } j \notin B \text{ or if } B\backslash j \in \mathcal{F} \\ B\backslash j & \text{if } j \in B \text{ and } B\backslash j \notin \mathcal{F} \end{cases}.$$

Then

$$T_j(\mathcal{F}) = \{T_j(B) \,:\, B \in \mathcal{F}\}.$$

We can repeatedly apply $T_j$ for each $j = 1, 2, \ldots, m$ to obtain the *shifted family* $T(\mathcal{F})$ which has the property that

$$T_j(T(\mathcal{F})) = T(\mathcal{F}) \text{ for } j = 1, 2, \ldots m.$$

Thus $|T(\mathcal{F})| = |\mathcal{F}|$ and $T(\mathcal{F})$ is a *downset* (namely for every $B \in T(\mathcal{F})$ and every $C \subseteq B$, we have $C \in T(\mathcal{F})$). Now let $S \subseteq [m]$ and let

$$\mathcal{F}|_S = \{B \cap S \,:\, B \in \mathcal{F}\}$$

**Theorem 10.1** *Let $S \subseteq [m]$.*

$$\left|\mathcal{F}|_S\right| \geq \left|T(\mathcal{F})|_S\right| \blacksquare$$

using this one can prove Theorem 1.3 by noting that if $\mathcal{F}$ has no configuration $K_k$ then for any $S \subseteq [m]$ with $|S| = k$, we have $|\mathcal{F}|_S| \leq 2^k - 1$ and hence $|T(\mathcal{F})|_S| \leq 2^k - 1$. Since $T(\mathcal{F})$ is a downset, the column of $k$ 1's is absent. Thus we deduce $|\mathcal{F}| = |T(\mathcal{F})| \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$ and hence prove Theorem 1.3.

Another application is for the forbidden matrix $F_3$ of Section 4, for which we note that a simple matrix $A$ avoiding $F_3$ has at most 6 column types on any 3 rows. The consequence is the exact bound of Theorem 4.3.

The shifting argument was utilized in [1] to obtain a forbidden configuration theorem associated with any ideal (downset) in the lattice of divisors. This led to the notion of order shattered sets in [30]. These lead to multiset versions of Theorem 10.1.

# 11 Graph Theory

The use of Graph Theory is easiest to understand in considering a $2 \times \ell$ forbidden configuration $F$. In that case, it is natural to form a graph whose vertices are the rows of the matrix $A$. We consider what is missing if we forbid a 2-rowed $F$ as in Section 8 and so columns in 'short supply' or absent can be noted in the graph perhaps using edge labels or directed edges (there are only 4 possible columns on 2 rows!). Results in that direction are repeatedly used in [24],[19],[27].

The Graph Theory theorems used include standard results about cliques, connectivity. The following specialized result was obtained in [19] to get one of the exact bounds in Table 1.

**Lemma 11.1** *Let $D = (N, A)$ be a directed graph. There is an ordering of the vertices $N$ as $1, 2, \ldots m$ where $m = |N|$ and a subset $T \subseteq A$ consisting of a collection of vertex disjoint indirected trees $T$ with the following property. Let $D_i$ denote the subgraph of $D$ induced by the vertices $\{i, i+1, \ldots m\}$. For each pair $i, j$, $1 \leq i < j \leq m$ either there is a directed path in $D_i$ from $i$ to $j$ or there is a $k$ with $i \leq k \leq m$ so that there is a directed path from $i$ to $k$ in $D_i$ and there is no edge in $D$ from $k$ to $j$.* $\blacksquare$

This is a generalization of Redei's Theorem that asserts that a tournament has a directed Hamiltonian path.

Graph Theory was successfully employed for larger $F$ in [32]. The decomposition of a directed graph into strongly connected components and an acyclic graph between the strongly connected components was an essential tool. We used that a linear number (linear in the number of vertices) of directed edges is sufficient to assure strong connectivity. This idea was again employed in [21] with indicator polynomials to establish Theorem 1.11.

## 12 Linear Algebra

Applications of linear algebra here include the proof of the Sauer Bound. Frankl and Pach obtained results for *null t-designs* [38]. One approach is the following. Given two columns $\beta, \gamma$, we say $\beta$ *covers* $\gamma$ if and only if $\beta \geq \gamma$. For an $m \times n$ simple matrix $A$ and an $m \times 1$ (0,1)-vector $\gamma$, we can define $A(\gamma)$ as the $1 \times n$ (0,1)-row vector with a 1 in position $j$ if column $j$ of $A$ covers $\gamma$.

Now the vector space $V = \text{span}\{A(\gamma) : \gamma \in \mathbf{R}^n\}$ is a vector space of dimension $n$ and moreover $\{A(\gamma) : \gamma \text{ is a column of } A\}$ is a basis for $V$. Now if we take

$$\Gamma_{k-1} = \{\gamma : \text{ there exists an } s \text{ with } 0 \leq s \leq k-1 \text{ and } \gamma \text{ is a column of } K_m^s\}$$

we are able to verify the following.

**Theorem 12.1** *[38]([46],[9]). let $A$ be an $m$-rowed simple matrix with no configuration $K_k$. Then $n$ is the dimension of $V = span\{A(\gamma) : \gamma \in \Gamma_{k-1}\}$ for $\Gamma_{k-1} = \{\gamma :$ there exists an $s$ with $0 \leq s \leq k-1$ and $\gamma$ is a column of $K_m^s\}$. Hence*

$$n \leq |\Gamma_{k-1}| = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \blacksquare$$

Another application of linear algebra is to considering columns in short supply [22]. The idea of indicator polynomials was further exploited in [21].

## 13 Strong Stability

The idea is to show that a set system satisfying some property (in our case having a forbidden configuration $F$) and having a number of sets close to the optimal value (for us $\text{forb}(m, F)$) that the system of sets has much of its structure already determined. This contrasts sharply with the Sauer Bound result for which there are a multiplicity of matrices achieving the given bound (e.g. Theorem 1.1[7], Theorem 4.2[10], Theorem 3.1[31]).

The strong stability result used in proving Theorem 6.2 considers a $k$-uniform set system with no $F_{0,r+1,r+1,0}$ (the notation $F_{abcd}$ is defined in Section 6) which is equivalent to having the set system be $k-r$-intersecting. Let numbers $k, r_1, r_2$ be given and suppose $G$ and $H$ are given disjoint sets with $|G| = k - r_1 + r_2$. We define $\mathcal{I}^k_{r_1,r_2}$ on the pair $(H, G)$ to be the family consisting of all sets of size $k$ in $G \cup H$ that intersect $G$ in at least $k - r_1 = |G| - r_2$ points. Note that any two sets in $\mathcal{I}^k_{r_1,r_2}$ have at least $|G| - 2r_2 = k - r_1 - r_2$ points in common, i.e. $\mathcal{I}^k_{r_1,r_2}$ is $(k - r)$-intersecting, where $r = r_1 + r_2$. The Complete Intersection Theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [2], is that any $k$-uniform, $(k-r)$-intersecting family of maximum size on a given ground set is isomorphic to $\mathcal{I}^k_{r-p,p}$, for some $0 \leq p \leq r$, which depends on the size of the ground set. Note that $|\mathcal{I}^k_{r_1,r_2}|$ is $O(m^r)$ ($\Theta(m^r)$ for $|G|$ and $|H|$ being $\Theta(m)$). We prove the following result.

**Theorem 13.1** *Suppose $\mathcal{A}$ is a $k$-uniform $(k-r)$-intersecting set system on $[m]$ of size at least $(5r)^{5r}m^{r-1}$. Then $\mathcal{A} \subseteq \mathcal{I}^k_{r-p,p}$ for some $0 \le p \le r$.*

We are also interested in the related family of sets $\mathcal{F}^k_{r_1,r_2}$ on the pair $(H,G)$ to be the family consisting of all sets of size $k$ in $G \cup H$ that intersect $G$ in at least $k - r_1 = |G| - r_2$ points. Note that $|\mathcal{F}^k_{r_1,r_2}|$ is also $O(m^r)$ and that $|\mathcal{I}^k_{r_1,r_2} \backslash \mathcal{F}^k_{r_1,r_2}|$ is $O(m^{r-2})$. A proof of Theorem 13.1 in the case $r = 1$ (where there are no aymptotics) is used in [14].

**Problem 13.2** *Can you to use Theorem 13.1 to prove some more exact bounds for $F$ being the $2k \times 2$ matrix of $k$ copies of $I_2$ for which the bound is $O(m^k)$ by Theorem 6.2? Theorem 5.6 is the case $k = 2$.*

# 14 Variations including Forbidden Submatrices

**Families of forbidden configurations**

There are many natural variations to the problem of forbidding a single configuration that have arisen. One possibility is to consider forbidding more than one configuration and some results are given below. Forbidding more than one configuration can have rather unpredictable consequences.

**Theorem 14.1** *(Balogh and Bollobás [33]) Let $k$ be given. Then forb$(m, \{I_k, I_k^c, T_k\})$ is $O(1)$.*

Besides reinforcing the interest in the building blocks of our conjecture, the result is also somewhat in line with a meta version of the conjecture which in general is false (Theorem 14.7 below). With Laura Dunwoody, we can establish exact results.

**Theorem 14.2** *[17] forb$(m, \{I_1, I_1^c, T_1\}) = 0$ and forb$(m, \{I_2, I_2^c, T_2\}) = 2$, forb$(m, \{I_3, I_3^c, T_3\}) = 6$.* ■

Any easy, but not optimal construction, of an $m \times \binom{2k}{k}$ simple matrix $A$ that has no configurations $I_k$, $I_k^c$, $T_k$ is to take all columns of column sum $k-1$ in the $(k-1)$-fold product $T_{m/(k-1)} \times T_{m/(k-1)} \times \cdots T_{m/(k-1)}$. One general result of Fleming with a good bound can be viewed as a special case:

**Theorem 14.3** *[20]Let $F_a = \begin{bmatrix} 1 & 0 & t \cdot 1 \\ 0 & 1 & t \cdot 1 \end{bmatrix}$, $F_b = \begin{bmatrix} 1 & 0 & t \cdot 0 \\ 0 & 1 & t \cdot 0 \end{bmatrix}$, and $F_c = t \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then for $t \ge 2$, forb$(m, \{F_a, F_b, F_c\}) \le 6t - 6$.*

**Theorem 14.4** *forb$(m, \{I_k, I_k^c\})$ is $\Theta(m^{k-1})$*

**Proof:** . We note that forb$(m, \{I_k\})$ is $O(m^{k-1})$ and hence forb$(m, \{I_k, I_k^c\})$ is $O(m^{k-1})$. The construction of the $(k-1)$-fold product $T_{m/(k-1)} \times T_{m/(k-1)} \times T_{m/(k-1)} \cdots \times T_{m/(k-1)}$ show that forb$(m, \{I_k, I_k^c\})$ is $\Omega(m^{k-1})$ since if we take two rows from any one term of the product, we are unable to have $I_2$ and yet $I_k$ and $I_k^c$ have $I_2$ in every pair of rows. ■

**Theorem 14.5** *forb*$(m,\{I_k^c,T_k\})$ *is* $\Theta(m^{k-1})$

**Proof:** . We note that forb$(m,\{I_k^c\})$ is $O(m^{k-1})$ and hence forb$(m,\{I_k^c,T_k\})$ is $O(m^{k-1})$. The construction of the $(k-1)$-fold product $I_{m/(k-1)}\times I_{m/(k-1)}\times I_{m/(k-1)}\cdots\times I_{m/(k-1)}$ show that forb$(m,\{I_k^c,T_k\})$ is $\Omega(m^{k-1})$ since if we take two rows from any one term of the product, we are unable to have $\begin{pmatrix}1\\1\end{pmatrix}$ and yet $I_k^c$ and $T_k$ have $\begin{pmatrix}1\\1\end{pmatrix}$ ■

**Theorem 14.6** *forb*$(m,\{I_k,T_k\})$ *is* $\Theta(m^{k-2})$

**Proof:** . We note that both $I_k$ and $T_k$ have a column with $k-1$ 0's and so neither can be found in the $(k-2)$-fold product $I^c_{m/(k-2)}\times I_{m/(k-2)}\times I^c_{m/(k-2)}\cdots\times I^c_{m/(k-2)}$ which shows that forb$(m,\{I_k,T_k\})$ is $\Omega(m^{k-2})$. To prove the upper bound we use induction on $\ell$ in the statement forb$(m,\{[K^0_{k-1}\,|\,I_{k-1}],T_\ell\})$ is $O(m^{\ell-2})$, for $\ell\ge 2$. When $\ell=2$, we note that forbidding $T_2$ means that any two sets that overlap must be disjoint. Then the condition no configuration $I_k$ means that there are at most $k-1$ disjoint nonempty sets and the empty set (the column of 0's). Thus forb$(m,\{I_k,T_2\})=k$. Also we have forb$(m,\{[K^0_{k-1}\,|\,I_{k-1}],T_2\})=k$. Now we use induction on $\ell$ and the standard decomposition of (37) to obtain that $B_2$ has no configurations $[K^0_{k-1}\,|\,I_{k-2}]$ or $T_{l-1}$. Applying induction, we obtain the desired bound. ■

The following result shows that our constructions of Conjecture 2.2 are no longer sufficient for asymptotics. General forbidden subgraph problems could be given this way.

**Theorem 14.7** *Let $C_4$ denote the $4\times 4$ matrix that is the incidence matrix of a cycle of length 4. Then forb$(m,\{K_3^3,C_4\})$ is $\Theta(m^{3/2})$.*

**Proof:** The act of forbidding $K_3^3$ is essentially making this into a graph problem. Apart from at most $m+1$ columns, all remaining columns must have two 1's and hence can be interpreted as edges of a graph on $m$ vertices. ■

Assume $t$ is given. Kleitman considered the maximum size of a set system $\mathcal{F}\subseteq 2^{[m]}$ with the property that for every pair $A,B\in\mathcal{F}$, $|A\backslash B|+|B\backslash A|\le 2t$. The bound is forb$(m,K_{t+1})$. One can think of this as having forbidden the $(2t+1)\times 2$ configurations $F_{0,2t+1,0,0},F_{0,2t,1,0},\ldots,F_{0,t+1,t,0}$.

Balanced and Totally Balanced matrices are easily defined in terms of forbidden configurations. Let $C_k$ denote the $k\times k$ matrix that is the incidence matrix of a cycle of length $k$. A matrix is *Balanced* if and only if it has no configuration $C_{2k+1}$ for $k=1,2,3,\ldots$. A matrix is *Totally Balanced* if and only if it has no configuration $C_k$ for $k=1,2,3,\ldots$. The result that forb$(m,C_3)=\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$ can be found in [46] but also follows from Theorem 1.3 since $C_3$ is a configuration of $K_3$.

**Theorem 14.8** *[18] Let $C_k$ denote the $k\times k$ matrix that is the incidence matrix of a cycle of length $k$. Then forb$(m,\{C_3,C_4,C_5,\ldots\})=$ forb$(m,C_3)=\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$.*

One has the remarkable result that any $m \times (\binom{m}{2} + \binom{m}{1} + \binom{m}{0})$ simple matrix with no configuration $C_3$ is also totally balanced (Remark 3.1[4]). Totally balanced matrices have been studied in many papers such as [18].

**Forbidden Submatrices**

Another variation is to ask whether the row or column order is critical. In most combinatorial investigations, permuting the row and column order may be little more than a relabelling. In other circumstances either the row order or the column order or both may be crucial, for example, in an algorithmic investigation where the algorithm proceeds by assuming you have a special ordering and then the algorithm exploits this special ordering [18]. It is a somewhat remarkable fact (due to Hoffman, Kolen and Sakarovitch [42] as well as [18]) that a matrix is Totally Balanced if and only if the rows and columns can be ordered so that the resulting matrix has no submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus in the case that row and column order both matter, we have the problem of forbidden submatrices. Results can be found in [9], [23], [37], [13].

**Theorem 14.9** *[13] Let $F$ be a $k \times \ell$ (0,1)-matrix. Let $A$ be an $m \times n$ (0,1)-matrix with no $k \times \ell$ submatrix of $A$ being equal to $F$. Then*

$$n \leq m^{2k-1-((k-1)/(13\log_2 \ell))}$$

This was an improvement on the original result that $n \leq m^{2k-1}$ proved in [37] via a pigeonhole argument (the first bound was $n \leq m^{13k\log_2 \ell}$ in [9]) . In any event the conjecture was made both in [23], [37] that:

**Conjecture 14.10** *[13] Let $F$ be a $k \times \ell$ (0,1)-matrix. Let $A$ be an $m \times n$ (0,1)-matrix with no $k \times \ell$ submatrix of $A$ being equal to $F$. Then there exists a constant $c_F$ depending only on $F$ so that*

$$n \leq c_F m^k.$$

**Fixed Row order**

There have been some investigations for cases where only column permutations are allowed. Some linear algebra proofs have this as an essential character [12]. Some induction proofs generalize, using this idea, to the idea of *order shattered sets* [30].

**Forbidding configurations on some selection of row subsets**

There are cases where one might want to forbid a configuration of $k$ rows on only some subset of the possible $k$-sets of rows. Induction, shifting and linear algebra proofs continue to work. A major result is that of Alon [3] and an exploration of the proof techniques and some generalizations are in [10]. An application of the result is in [13] to the problem of forbidden submatrices.

The main results on shattered sets are stated from a different point of view (typically assuming some configurations are present on certain subsets of the rows) but are related (e.g. [30]).

**Multiset versions**

Many results easily extend to multisets, the usual approach being to allow element $i$ (corresponding to row $i$) to have maximum multiplicity $e_i$. The extension of Theorem 1.3 to multisets with $e_1 = e_2 = \cdots = e_m = e$ is in [44], the extension to forbidding $K_{|S|}$ on rows $S$ for a family of sets $S \in T \subseteq 2^{[m]}$ while having various element multiplicities is in [3]. Define an $m$-rowed matrix $A$ to be $e$-simple if there are no repeated columns and if any entry in the $i$th row of $A$ is chosen from $\{0, 1, \ldots, e_i\}$ for $i = 1, 2, \ldots, m$. In this context, we use $K_S$ to denote the $k \times \left(\prod_{i \in S}(e_i + 1)\right)$ $e$-simple matrix.

**Theorem 14.11** *[3]. Let $m, e_1, e_2, \ldots, e_m$ be given positive integers and let $T$ be given with $T \subseteq 2^{[m]}$. Let $f(T)$ be the number of $(m, e_1, e_2, \ldots, e_m)$-columns which do not have all 0's for the rows indexed by $S$ for any $S \in T$. Then if $A$ is $m \times n$ e-simple matrix with no configuration $K_S$ for any $S \in T$, then*

$$n \leq f(T). \blacksquare$$

There are some forbidden configuration ideas in [28] that explore the natural generalization of $K_k^s$ and Theorem 1.6 to the multisets. The results in [1] are stated in terms of divisors of an integer $\prod_{i=1}^{m} p_i^{e_i}$.

Interestingly, the Bixby and Cunningham proof of the bound on the number of distinct columns for a totally unimodular matrix, a (-1,0,1)-matrix, uses the Sauer Bound of Theorem 1.3. Further applications to matrices with more than just two possible entries are found in [11].

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