Forbidden Configurations

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What is an Extremal Problem?
Here are some examples of extremal problems:

- At most how many queens can we place on a chessboard so that no two attack each other? Answer: 8
- At most how many $2 \times 1$ dominoes can we place on a chessboard which has two opposite corners removed? Answer: 30
- At most how many edges can a simple graph with $p$ vertices have, if it has no triangles? Answer: $\lfloor \frac{p^2}{4} \rfloor$ (Turán's Theorem)
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Definition. A simple matrix is a \(\{0,1\}\)-matrix with no repeated columns.
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\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
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0 & 0 & 1 & 1 & 0 & 1 \\
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\end{bmatrix}
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e.g. 

We can think of an \( m \)-rowed simple matrix as the incidence matrix of a collection of subsets of \( \{1, 2, \ldots, m\} \).
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\( \emptyset \quad \{1\} \quad \{1,3,4\} \quad \{3,4\} \quad \{2,4\} \quad \{2,3,4\} \)

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We can think of an \(m\)-rowed simple matrix as the incidence matrix of a collection of subsets of \(\{1,2,\ldots,m\}\).

An \(m\)-rowed simple matrix has at most \(2^m\) columns.
**Definition.** A simple matrix is a \(\{0,1\}\)-matrix with no repeated columns.

**Definition.** Suppose that \(F\) is a \(\{0,1\}\)-matrix (not necessarily simple). A simple matrix \(A\) has the configuration \(F\) if \(A\) has a submatrix which is a row and column permutation of \(F\).
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**Extremal Problem:** If a simple matrix $A$ has $m$ rows and does not have the configuration $F$, at most how many columns can $A$ have?

**Answer:** $\text{forb}(m, F)$
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Definition. Suppose that \( F \) is a \( \{0,1\} \)-matrix, and \( m \) a positive integer. Then \( \text{forb}(m, F) \) is the greatest number of columns that an \( m \)-rowed simple matrix with no configuration \( F \) can have.
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Equivalently, \(\text{forb}(m, F)\) is the least integer such that every simple matrix with \(m\) rows and more than \(\text{forb}(m, F)\) columns has the configuration \(F\).
Examples

\[ \text{forb}(m, [1 \ 0]) = 1. \]
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\text{forb}(m, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}) = 2m + 2.
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\[ \text{forb}(m, \begin{bmatrix} 1 & 0 \end{bmatrix}) = 1. \]

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Some New Results for 2-Columnned F

Others proved previously that

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1,$$

What happens if we keep adding $\begin{bmatrix} 1 & 0 \end{bmatrix}$ on top?
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$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \left\lceil \frac{3}{2}m \right\rceil + 1,$$

∀ \text{ } m \geq 3.
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\[ \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{2} + m + 2 \quad \forall m \geq 3. \]
Some New Results for 2-Columned F

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\]

What happens if we keep adding \([1 \ 0]\) on top?
**Theorem.** For $m \geq 3$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{2} + m + 2.$$ 

A construction: the $m$-rowed matrix with all columns of sum $0, 1, 2$ and $m$. 
Theorem. For $m \geq 4$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{3} + \binom{m}{2} + m + 2.$$ 

A construction: the $m$-rowed matrix with all columns of sum $0, 1, 2, 3$ and $m$. 
Theorem. For $m \geq 5$, 

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{4} + \binom{m}{3} + \binom{m}{2} + m + 2.$$ 

A construction: the $m$-rowed matrix with all columns of sum 

$0, 1, 2, 3, 4$ and $m$. 

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Theorem. For \( m \geq k - 1 \geq 3 \),

\[
\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{k-2} + \cdots + \binom{m}{2} + m + 2.
\]

A construction: the \( m \)-rowed matrix with all columns of sum

\( 0, 1, 2, \ldots, k - 2 \) and \( m \).
**Theorem.** For $m \geq k - 1 \geq 3$,\

$$\text{forb}(m, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}^k) = \binom{m}{k-2} + \cdots + \binom{m}{2} + m + 2.$$\

A construction: the $m$-rowed matrix with all columns of sum $0, 1, 2, \ldots, k - 2$ and $m$. 
I asked, “What if I flip some digits in the second column?”

**Theorem.** For $m \geq k - 1 \geq 3$, $m \geq k - 2 \geq 3$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{k-2} + \cdots + \binom{m}{2} + m + 2.$$ 

A construction: the $m$-rowed matrix with all columns of sum $0, 1, 2, \ldots, k - 2$ and $m$. 
The bound and the construction remains the same!

**New! Theorem.** For $m \geq k - 1 \geq 3$, 

\[
\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) \{ k \} ) = \binom{m}{k-2} + \cdots + \binom{m}{2} + m + 2.
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\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T, k) = \binom{m}{k - 2} + \cdots + \binom{m}{2} + m + 2.
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If both columns have $k - 1$ ones, then strange things happen.

For $m \geq k - 1 \geq 3$,

$$\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \text{KABOOM}$$
If both columns have $k - 1$ ones, then strange things happen.

For $m \geq k - 1 \geq 3$,

$$\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, k) = \text{KABOOM}$$

Finding a good construction becomes a difficult Design Theory problem.
What if we flip the 1 at the bottom of the second column to a 0?

For $m \geq k - 1 \geq 3$, 

$$\text{forb}(m, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cup k) = \text{KABOOM}.$$
We get the same result as before.

**New! Theorem.** For $m \geq k - 1 \geq 3$,

\[
\text{forb}(m, \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} k \end{bmatrix}) = \binom{m}{k-2} + \cdots + \binom{m}{2} + m + 2.
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A construction: the $m$-rowed matrix with all columns of sum $0, 1, 2, \ldots, k - 2$ and $m$. 
**Sketch of Induction Proof:**
For a given $F$, let $A$ be a simple $m \times \text{forb}(m, F)$ matrix which does not have the configuration $F$. 

We permute the columns of $A$ so that it looks like
\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\end{bmatrix}
\]
where $D$ is the matrix of columns repeated under the first row. Then $C, D, E$ concatenated together is simple and $(m - 1)$-rowed, and does not have the configuration $F$.

\[\#\text{col's}(A) = \#\text{col's}(C, D, E) + \#\text{col's}(D)\]
\[\text{forb}(m, F) \leq \text{forb}(m - 1, F) + \#\text{col's}(D)\]

If we can get a good upper bound on $\#\text{col's}(D)$, then we can prove an upper bound on $\text{forb}(m, F)$ by induction.
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\end{bmatrix},
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where $D$ is the matrix of columns repeated under the first row.

Then $C, D, E$ concatenated together is simple and $(m - 1)$-rowed, and does not have the configuration $F$. 

$$
\therefore \text{col}'s(A) = \text{col}'s(C, D, E) + \text{col}'s(D) \leq \text{forb}(m, F) + \text{col}'s(D)
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If we can get a good upper bound on $\#\text{col's}(D)$, then we can prove an upper bound on $\text{forb}(m, F)$ by induction.
Thank You!

Thanks for your attention!