Forbidden Families of Configurations

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Joint work with Christina Koch
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Consider the following family of subsets of \( \{1, 2, 3, 4\} \):
\[
\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}
\]
The incidence matrix \( A \) of the family \( \mathcal{A} \) of subsets of \( \{1, 2, 3, 4\} \) is:
\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

**Definition** We say that a matrix \( A \) is simple if it is a (0,1)-matrix with no repeated columns.

**Definition** We define \( \|A\| \) to be the number of columns in \( A \).
\[
\|A\| = 6 = |\mathcal{A}|
\]
**Definition** Given a matrix $F$, we say that $A$ has $F$ as a configuration (denoted $F \prec A$) if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \prec \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$
**Definition** Given a matrix \( F \), we say that \( A \) has \( F \) as a *configuration* (denoted \( F \prec A \)) if there is a submatrix of \( A \) which is a row and column permutation of \( F \).

\[
F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \prec A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}
\]

**Definitions**

\( \mathcal{F} = \{F_1, F_2, \ldots, F_t\} \)

\( \text{Avoid}(m, \mathcal{F}) = \{A : A \text{ } m \text{-rowed simple, } F \not\prec A \text{ for all } F \in \mathcal{F}\} \)

\( \text{forb}(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\} \)
**Definition** Let $K_k$ be the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

**Theorem** (Füredi 83). Let $F$ be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.

**Problem** Given $F$, can we predict the behaviour of $forb(m, F)$?
Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

\[ C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \]
Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

$$e.g. \quad C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Matrices in $\text{Avoid}(m, \{ C_3, C_5, C_7, \ldots \})$ are called **Balanced Matrices**.

**Theorem** \( \text{forb}(m, \{ C_3, C_5, C_7, \ldots \}) = \text{forb}(m, C_3) \)
Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

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\]

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**Theorem** $\text{forb}(m, \{C_3, C_5, C_7, \ldots\}) = \text{forb}(m, C_3)$

Matrices in $\text{Avoid}(m, \{C_3, C_4, C_5, C_6, \ldots\})$ are called Totally Balanced Matrices.

**Theorem** $\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \text{forb}(m, C_3)$
The inequality $\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) \leq \text{forb}(m, C_3)$ follows from the following:

**Lemma** If $\mathcal{F}' \subset \mathcal{F}$ then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{F}')$.

The equality follows from a result that any $m \times \text{forb}(m, C_3)$ simple matrix is in fact totally balanced (A, 80). Thus we conclude

$\text{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \text{forb}(m, C_3)$. 

A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
**Definition** Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \times \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
The Conjecture

**Definition** Let $x(F)$ denote the smallest $p$ such that every $p$-fold product contains $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{c m/p}, T_{m/p}\}$. 
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**Conjecture** (A, Sali 05) $\text{forb}(m, F)$ is $\Theta(m^{x(F)-1})$.

In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.
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The conjecture has been verified for $k \times \ell \ F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.
**Definition** \(\text{ex}(m, H)\) is the maximum number of edges in a (simple) graph \(G\) on \(m\) vertices that has no subgraph \(H\).

A \(\in\) Avoid\((m, 1_3)\) will be a matrix with up to \(m + 1\) columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.

Let \(I(H)\) denote the \(|V(H)| \times |E(H)|\) vertex-edge incidence matrix associated with \(H\).

**Theorem** \(\text{forb}(m, \{1_3, I(H)\}) = m + 1 + \text{ex}(m, H)\).
Definition \( \text{ex}(m, H) \) is the maximum number of edges in a (simple) graph \( G \) on \( m \) vertices that has no subgraph \( H \).

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Let \( I(H) \) denote the \( |V(H)| \times |E(H)| \) vertex-edge incidence matrix associated with \( H \).

**Theorem** \( \text{forb}(m, \{1_3, I(H)\}) = m + 1 + \text{ex}(m, H) \).

In this talk \( I(C_4) = C_4 \), \( I(C_6) = C_6 \).

**Theorem** \( \text{forb}(m, \{1_3, C_4\}) = m + 1 + \text{ex}(m, C_4) \) which is \( \Theta(m^{3/2}) \).

**Theorem** \( \text{forb}(m, \{1_3, C_6\}) = m + 1 + \text{ex}(m, C_6) \) which is \( \Theta(m^{4/3}) \).
**Theorem** (Balogh and Bollobás 05) Let $k$ be given. Then there is a constant $c_k$ so that $\text{forb}(m, \{I_k, I_k^c, T_k\}) = c_k$.

We note that there is no obvious product construction.

Note that $c_k \geq \binom{2k-2}{k-1}$ by taking all columns of column sum at most $k - 1$ that arise from the $k - 1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$.
Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

**Lemma** Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i$, there is some $F_j$ with $F_j \prec G_i$. Then $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{G})$. 
Let \( \mathcal{F} = \{F_1, F_2, \ldots, F_k\} \) and \( \mathcal{G} = \{G_1, G_2, \ldots, G_\ell\} \).

**Lemma** Let \( \mathcal{F} \) and \( \mathcal{G} \) have the property that for every \( G_i \), there is some \( F_j \) with \( F_j \prec G_i \). Then \( \text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{G}) \).

**Theorem** Let \( \mathcal{F} \) be given. Then either there is a constant \( c \) with \( \text{forb}(m, \mathcal{F}) = c \) or \( \text{forb}(m, \mathcal{F}) \) is \( \Omega(m) \).
Let \( F = \{ F_1, F_2, \ldots, F_k \} \) and \( G = \{ G_1, G_2, \ldots, G_\ell \} \).

**Lemma** Let \( F \) and \( G \) have the property that for every \( G_i \), there is some \( F_j \) with \( F_j \prec G_i \). Then \( \text{forb}(m, F) \leq \text{forb}(m, G) \).

**Theorem** Let \( F \) be given. Then either there is a constant \( c \) with \( \text{forb}(m, F) = c \) or \( \text{forb}(m, F) \) is \( \Omega(m) \).

**Proof:** We start using \( G = \{ I_p, I_p^c, T_p \} \) with \( p \) suitably large. Either we have the property that there is some \( F_r \prec I_p \), and some \( F_s \prec I_p^c \) and some \( F_t \prec T_p \) in which case \( \text{forb}(m, F) \leq \text{forb}(m, \{ I_p, I_p^c, T_p \}) = O(1) \) or without loss of generality we have \( F_j \not\prec I_p \) for all \( j \) and hence \( I_m \in \text{Avoid}(m, F) \) and so \( \text{forb}(m, F) \) is \( \Omega(m) \).
A pair of Configurations with quadratic bounds

e.g. \( F_2(1, 2, 2, 1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \not\in I \times I^c. \)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
=\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
A pair of Configurations with quadratic bounds

e.g.  $F_2(1, 2, 2, 1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \notin I \times I^c$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{l_3} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{l_3^c} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}_{I_{m/2} \times I_{m/2}^c}$$

is an $m \times m^2/4$ simple matrix avoiding $F_2(1, 2, 2, 1)$, so $\text{forb}(m, F_2(1, 2, 2, 1))$ is $\Omega(m^2)$.

(A, Ferguson, Sali 01  $\text{forb}(m, F_2(1, 2, 2, 1)) = \lfloor \frac{m^2}{4} \rfloor + \binom{m}{1} + \binom{m}{0}$)
A pair of Configurations with quadratic bounds

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin T \times T$.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{T_3} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{T_3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$
A pair of Configurations with quadratic bounds

e.g. \( I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin T \times T. \)

\[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\]

\( T_{m/2} \times T_{m/2} \) is an \( m \times m^2/4 \) simple matrix avoiding \( I_3 \), so \( \text{forb}(m, I_3) \) is \( \Omega(m^2) \).

\( (\text{forb}(m, I_3) = \begin{pmatrix} m \\ 2 \end{pmatrix} + \begin{pmatrix} m \\ 1 \end{pmatrix} + \begin{pmatrix} m \\ 0 \end{pmatrix} ) \)
By considering the construction $I \times I^c$ that avoids $F_2(1, 2, 2, 1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids $I_3$, we note that we have only linear obvious constructions ($I^c_m$ or $T_m$) that avoid both $F_2(1, 2, 2, 1)$ and $I_3$. We are led to the following: Theorem $forb(m, \{I_3, F_2(1, 2, 2, 1)\})$ is $\Theta(m)$. 
By considering the construction $I \times I^c$ that avoids $F_2(1, 2, 2, 1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids $I_3$, we note that we have only linear obvious constructions ($I_m^c$ or $T_m$) that avoid both $F_2(1, 2, 2, 1)$ and $I_3$. We are led to the following:

**Theorem** $\text{forb}(m, \{I_3, F_2(1, 2, 2, 1)\})$ is $\Theta(m)$.

We can extend the argument quite far:

**Theorem** $\text{forb}(m, \{t \cdot I_k, F_2(1, t, t, 1)\})$ is $\Theta(m)$. 
By considering the construction $I \times I_c$ that avoids $F_2(1, 2, 2, 1)$ and the constructions $I_c \times I_c$ or $I_c \times T$ or $T \times T$ that avoids $l_3$, we note that we have only linear obvious constructions ($I_m^c$ or $T_m$) that avoid both $F_2(1, 2, 2, 1)$ and $l_3$. We are led to the following:

**Theorem** $\text{forb}(m, \{l_3, F_2(1, 2, 2, 1)\})$ is $\Theta(m)$.

We can extend the argument quite far:

**Theorem** $\text{forb}(m, \{t \cdot l_k, F_2(1, t, t, 1)\})$ is $\Theta(m)$.

We studied the 9 ‘minimal’ configurations that have quadratic bounds and were able to verify the predictions of the conjecture for all pairs. A variety of proofs of the upper bounds were employed.
Using our standard induction one can prove the following.

**Theorem** Let $k$ be given. Then $\text{forb}(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$ is $\Theta(m)$.

$I_m \in \text{Avoid}(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$. 
Using our standard induction one can prove the following.

**Theorem** Let $k$ be given. Then $forb(m, \{2 \cdot I_k, 2 \cdot I^c_k, 2 \cdot T_k\})$ is $\Theta(m)$.

$l_m \in Avoid(m, \{2 \cdot I_k, 2 \cdot I^c_k, 2 \cdot T_k\})$.

**Theorem** Let $k, t$ be given. Then $forb(m, \{t \cdot I_k, t \cdot I^c_k, t \cdot T_k\})$ is $\Theta(m)$. 
An unusual Bound

**Theorem** (A, Koch, Raggi, Sali 12) \( \text{forb}(m, \{T_2 \times T_2, I_2 \times I_2\}) \) is \( \Theta(m^{3/2}) \).

\[
T_2 \times T_2 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}, \quad I_2 \times I_2 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix} \quad (= C_4)
\]

We showed initially that \( \text{forb}(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\}) \) is \( \Theta(m^{3/2}) \) but Christina Koch realized that we ought to be able to drop \( T_2 \times I_2 \) and we were able to redo the proof (which simplified slightly!).
Forbidden Families of Configurations
Induction

Let $A$ be an $m \times \text{forb}(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{ T_2 \times T_2, I_2 \times I_2 \}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \text{row } r \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & C_r & C_r & D_r \end{bmatrix}.$$
Let $A$ be an $m \times \text{forb}(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{bmatrix}
\text{row } r & 0 & \cdots & 0 & 1 & \cdots & 1 \\
B_r & C_r & C_r & D_r
\end{bmatrix}.$$ 

To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of $r$. Our proof shows that assuming $\|C_r\| > 20m^{1/2}$ for all choices $r$ results in a contradiction. In particular, associated with $C_r$ is a set of rows $S(r)$ with $S(r) \geq 5m^{1/2}$. We let $S(r) = \{r_1, r_2, r_3, \ldots\}$. After some work we show that $|S(r_i) \cap S(r_j)| \leq 5$. Then we have

$$|S(r_1) \cup S(r_2) \cup S(r_3) \cup \cdots|$$

$$= |S(r_1)| + |S(r_2) \setminus S(r_1)| + |S(r_3) \setminus (S(r_1) \cup S(r_2))| + \cdots$$

$$= 5m^{1/2} + (5m^{1/2} - 5) + (5m^{1/2} - 10) + \cdots > m ! ! !$$
Thanks to Ryan Martin for the invite!
Great to see Ames, Iowa.