Forbidden Configurations

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The paper ‘Small Forbidden Configurations’, joint with Jerry Griggs and Attila Sali, began a systematic exploration of the subject. The collaboration is from a sabbatical visit of Jerry to Vancouver and a visit of Attila in 1993. That paper contains the origin of the conjecture that I will describe.

Survey at www.math.ubc.ca/~anstee
Being taught birdwatching by Jerry

Richard Anstee, UBC, Vancouver
Jerry isn’t tall
One birdwatcher and one pack animal
Jerry and Jeannine in Magnolia Gardens
Deynise, Malia and Jerry
**Definition** We say that a matrix $A$ is *simple* if it is a $(0,1)$-matrix with no repeated columns.
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i.e. if $A$ is $m$-rowed then $A$ is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$$
**Definition** Given a matrix $F$, we say that $A$ has $F$ as a configuration written $F \prec A$ if there is a submatrix of $A$ which is a row and column permutation of $F$. 

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \prec \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = A$$
Our Extremal Problem

Definition We define $\|A\|$ to be the number of columns in $A$.

$\text{Avoid}(m, \mathcal{F}) = \{ A : A \text{ is } m\text{-rowed simple, } F \not\preceq A \text{ for } F \in \mathcal{F} \}$
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$\text{forb}(m, \mathcal{F}) = \max_A \{ \|A\| : A \in \text{Avoid}(m, \mathcal{F}) \}$
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There are other possibilities for extremal problems for $\text{Avoid}(m, \mathcal{F})$ including maximizing the weighted sum over columns where a column of column sum $i$ is weighted by $1/(m_i)$ (e.g. Johnson and Lu) or maximizing the number of 1’s.
A Product Construction

As with any extremal problem, the results are often motivated by constructions, namely matrices in Avoid(m, F). Early investigations with Jerry Griggs and Attila Sali suggested a product construction might be very helpful.

The building blocks of our product constructions are $I$, $I_c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Definition  Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
A Product Construction

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.
Definition Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{c_{m/p}}^c, T_{m/p}\}$.
The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I_{m/p}^c, T_{m/p}\}$.

**Conjecture** (A, Sali 05) $\text{forb}(m, F)$ is $\Theta(m^{x(F)})$.

In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions. The conjecture has now been verified in many cases.
Definition Let $s \cdot F = \left[ F F \cdots F \right]$.

Let $F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Theorem (Frankl, Füredi, Pach 87) $\text{forb}(m, F) = \binom{m}{2} + 2m - 1$
i.e. $\text{forb}(m, F)$ is $\Theta(m^2)$.

Theorem (A. and Lu 13) Let $s$ be given. Then $\text{forb}(m, s \cdot F)$ is $\Theta(m^2)$.

Note $x(F) = 2 = x(s \cdot F)$ for a constant $s$, so this is evidence for the conjecture...
Berge Hypergraphs

Claude Berge, and others, created hypergraphs as a generalization of graphs. There are several hypergraph generalizations of paths and cycles. One generalization yields Berge paths and cycles. The definition of Berge Hypergraphs was given to me by Gerbner and Palmer (2015) and follows the same ideas. With Santiago Salazar, we consider the extremal set problem obtained by forbidding a single Berge Hypergraph.
Let $F$ be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph $H$ has $F$ as a Berge Hypergraph and write $F \ll H$ if there are $\ell$ edges $E'_1, E'_2, \ldots, E'_\ell$ of $H$ so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.

$$F = C_4$$
$$E_1 = \{1, 2\}$$
$$E_2 = \{2, 3\}$$
$$E_3 = \{3, 4\}$$
$$E_4 = \{1, 4\}$$
Berge Hypergraphs

Let $F$ be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph $H$ has $F$ as a Berge Hypergraph and write $F \preccurlyeq H$ if there are $\ell$ edges $E'_1, E'_2, \ldots, E'_\ell$ of $H$ so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.

\[ F = C_4 \preccurlyeq H \]

\[
E_1 = \{1, 2\} \quad E'_1 = \{1, 2, 4\} \\
E_2 = \{2, 3\} \quad E'_2 = \{2, 3, 5\} \\
E_3 = \{3, 4\} \quad E'_3 = \{3, 4\} \\
E_4 = \{1, 4\} \quad E'_4 = \{1, 3, 4, 5\} 
\]
Define our extremal problem as follows:

\[ \text{BergeAvoid}(m, F) = \{ A : A \text{ is } m\text{-rowed, simple, } F \not\preceq A \}, \]

\[ \text{Bforb}(m, F) = \max \{ \|A\| : A \in \text{BergeAvoid}(m, F) \}. \]
**Theorem** If $A \in \text{BergeAvoid}(m, F)$, then there exists an $A' \in \text{BergeAvoid}(m, F)$ with $\|A\| = \|A'\|$ and the columns of $A'$ form a downset: namely if $\alpha$ is a column of $A'$ and $\beta \leq \alpha$, then $\beta$ is also a column of $A'$.

**Proof:** Apply a shifting argument, replacing 1’s by 0’s in $A$ as long as no repeated columns are created. The result is $A'$.

**Theorem** $\text{Bforb}(m, I_k) = 2^{k-1}$
**Theorem** \( B_{\text{forb}}(m, C_4) = \Theta(m^{3/2}) \)

**Theorem** Let \( t \geq 3 \). Then \( B_{\text{forb}}(m, I_3 \times I_t) = \Theta(m^2) \)

For this latter result we needed recent extremal graph results. Note that \( I_3 \times I_t \) is the vertex-edge incidence matrix of \( K_{3,t} \).
**Theorem** \( \text{Bforb}(m, C_4) = \Theta(m^{3/2}) \)

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**Definition** \( \text{ex}(m, K_{\ell}, K_{s,t}) \) is the maximum number of copies of \( K_{\ell} \) in an \( m \)-vertex \( K_{s,t} \)-free graph.

Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

**Theorem** (Alon, Shikhelman '15, Kostachka, et al '15)
Let \( s, t \) be given with \( t \geq (s - 1)! + 1 \). Then \( \text{ex}(m, K_3, K_{s,t}) = \Theta(m^{3-(3/s)}) \).
Linyuan and his kids on Pender Island
**Theorem** (Balogh and Bollobás 05) Let $k$ be given. Then

\[ \text{forb}(m, \{I_k, I_k^c, T_k\}) \leq 2^{2^k} \]
An Unavoidable Forbidden Family

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**Theorem** (A., Lu 14) Let \( k \) be given. Then there is a constant \( c \)

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Theorem (A., Lu 14) Let $k$ be given. Then there is a constant $c$

$$\text{forb}(m, \{I_k, I_k^c, T_k\}) \leq 2^{ck^2}$$

If you take all columns of column sum at most $k - 1$ that arise from the $k - 1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$ then this yields $\binom{2k-2}{k-1} \approx 2^{2^k}$ columns. A probabilistic construction in $\text{Avoid}(m, \{I_k, I_k^c, T_k\})$ has $2^{ck \log k}$ columns.
Proofs used lots of induction and multicoloured Ramsey numbers: $R(k_1, k_2, \ldots, k_\ell)$ is the smallest value of $n$ such than any colouring of the edges of $K_n$ with $\ell$ colours 1, 2, \ldots, $\ell$ will have some colour $i$ and a clique of $k_i$ vertices with all edges of colour $i$. These numbers are readily bounded by multinomial coefficients:

$$R(k_1, k_2, \ldots, k_\ell) \leq \binom{\sum_{i=1}^{\ell} k_i}{k_1, k_2, k_3, \ldots, k_\ell}$$

$$R(k_1, k_2, \ldots, k_\ell) \leq \ell^{k_1+k_2+\ldots+k_\ell}$$

Our first proof had something like

$$\text{for} \text{b}(m\{, I_k, I_k^c, T_k\}) < R(R(k, k), R(k, k))$$

yielding a doubly exponential bound.
We say a matrix with entries in $\{0, 1, \ldots, r-1\}$ is an $r$-matrix. An $r$-matrix is simple if there are no repeated columns.

$$\text{forb}(m, r, \mathcal{F}) = \max\{\|A\| : A \text{ is simple } r\text{-matrix}, F \not\prec A \ \forall F \in \mathcal{F}\}$$
We say a matrix with entries in \( \{0, 1, \ldots, r - 1\} \) is an \( r \)-matrix. An \( r \)-matrix is simple if there are no repeated columns.

\[
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\]

Let \( T_k(a, b, c) = \begin{bmatrix}
  b & c & c & \cdots & c \\
  a & b & c & \cdots & c \\
  a & a & b & \cdots & c \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & a & a & \cdots & b
\end{bmatrix} \)

Let \( \mathcal{T}_k(r) = \{ T_k(a, b, c) : a \neq b, \ a, b, c \in \{0, 1, \ldots, r - 1\} \} \)
We say a matrix with entries in \(\{0, 1, \ldots, r - 1\}\) is an \(r\)-matrix. An \(r\)-matrix is simple if there are no repeated columns.

\[
\text{forb}(m, r, \mathcal{F}) = \max\{\|A\| : A \text{ is simple } r\text{-matrix}, F \not\prec A \ \forall F \in \mathcal{F}\}
\]

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b & c & c & \cdots & c \\
a & b & c & \cdots & c \\
a & a & b & \cdots & c \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a & a & a & \cdots & b
\end{bmatrix}\)

Let \(\mathcal{T}_k(r) = \{T_k(a, b, c) : a \neq b, \ a, b, c \in \{0, 1, \ldots, r - 1\}\}\)

**Theorem** (A, Lu 14) Given \(r\) there exists a constant \(c_r\) so that

\[
\text{forb}(m, r, \mathcal{T}_k(r)) \leq 2^{c_r k^2}.
\]
Consider 3-matrices, that is matrices with entries in \( \{0, 1, 2\} \). By Ramsey Theory, if \( n \geq R(k, k, k) \), then any choices for the entries marked \( * \) in the \( n \times n \) matrix

\[
\begin{bmatrix}
  b & * & * & \cdots & * \\
  a & b & * & \cdots & * \\
  a & a & b & \cdots & * \\
  \vdots & \vdots & \vdots & \ddots & \ddots \\
  a & a & a & \cdots & b
\end{bmatrix}
\]

we will find one of the configurations \( T_k(a, b, 0) \) or \( T_k(a, b, 1) \) or \( T_k(a, b, 2) \).
\[ T_k(2) = \]
\[
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & \ldots & 1 \\
1 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 0
\end{bmatrix},
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
0 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix}.\]
\( \mathcal{T}_k(3) \setminus \mathcal{T}_k(2) = \)
\[
\begin{bmatrix}
1 & 2 & \cdots & 2 \\
2 & 1 & \cdots & 2 \\
2 & 2 & \cdots & 1 \\
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
1 & 1 & \cdots & 2 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
2 & 0 & \cdots & 0 \\
2 & 2 & \cdots & 0 \\
0 & 2 & \cdots & 2 \\
2 & 0 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
0 & 2 & \cdots & 2 \\
0 & 2 & \cdots & 2 \\
2 & 0 & \cdots & 2 \\
2 & 2 & \cdots & 2 \\
0 & 2 & \cdots & 2 \\
0 & 0 & \cdots & 2 \\
0 & 0 & \cdots & 2 \\
1 & 2 & \cdots & 2 \\
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
2 & 0 & \cdots & 1 \\
2 & 2 & \cdots & 0 \\
2 & 2 & \cdots & 1 \\
\end{bmatrix}
\]

**Problem** Let \( \mathcal{F} \) be a family of \((0, 1)\)-matrices. Is it true that \( \text{forb}(m, 3, (\mathcal{T}_k(3) \setminus \mathcal{T}_k(2) \cup \mathcal{F})) \) is \( \Theta(\text{forb}(m, \mathcal{F})) \)?
Do the set of matrices in \( \text{BergeAvoid}(m, 3, (T_k(3) \setminus T_k(2)) \) behave somewhat like (0,1)-matrices?

\[
T_k(0, 2, 1) = \begin{bmatrix}
2 & 1 & 1 & \cdots & 1 \\
0 & 2 & 1 & \cdots & 1 \\
0 & 0 & 2 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2
\end{bmatrix}
\]

**Theorem** Let \( \mathcal{F} \) be a family of (0,1)-matrices. 

\[
\text{forb}(m, 3, (T_k(3) \setminus T_k(2) \cup T_k(0, 1, 2) \cup \mathcal{F})) \text{ is } \Theta(\text{forb}(m, \mathcal{F})).
\]

Surely \( T_k(0, 1, 2) \) is not needed for this result. Dawson, Lu, Sali and A. '17 have some preliminary results on eliminating \( T_k(0, 1, 2) \). Our results have made heavy use of Ramsey Theory.
**Corollary** Let $F \prec T_k(0, 2, 1)$. Then \(\text{forb}(m, 3, (T_k(3) \setminus T_k(2) \cup F))\) is \(\Theta(\text{forb}(m, F))\).

**Theorem** \(\text{forb}(m, 3, (T_k(3) \setminus T_k(2)))\) is \(\Theta(2^m)\) which is \(\Theta(\text{forb}(m, \emptyset))\).

**Theorem** \(\text{forb}(m, 3, (T_k(3) \setminus T_k(2) \cup l_2))\) is \(\Theta(\text{forb}(m, l_2))\).

A nice inductive result:

**Theorem** \(\text{forb}(m, 3, (T_k(3) \setminus T_k(2) \cup \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times F))\) is \(\Theta(m \cdot \text{forb}(m, 3, (T_k(3) \setminus T_k(2) \cup F)))\).
Congratulations on this milestone.
And thank you, Jerry, for your friendship over the years.