Claude Berge and others created hypergraphs as a generalization of graphs. There are several hypergraph generalizations of paths and cycles. One generalization yields Berge paths and cycles. The definition of Berge Hypergraphs was given by Gerbner and Palmer (2015) and follows the same ideas. We consider the extremal set problem obtained by forbidding a single Berge Hypergraph.
Let $F$ be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph $H$ has $F$ as a Berge Hypergraph and write $F \prec H$ if there are $\ell$ edges $E_1', E_2', \ldots, E_\ell'$ of $H$ so that $E_i \subseteq E_i'$ for $i = 1, 2, \ldots, \ell$.

$$F = C_4$$

$E_1 = \{1, 2\}$

$E_2 = \{2, 3\}$

$E_3 = \{3, 4\}$

$E_4 = \{1, 4\}$
Let $F$ be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph $H$ has $F$ as a Berge Hypergraph and write $F \preccurlyeq H$ if there are $\ell$ edges $E'_1, E'_2, \ldots, E'_\ell$ of $H$ so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.

![Diagram of hypergraphs]

$F = C_4 \preccurlyeq H$

$E_1 = \{1, 2\}$  $E'_1 = \{1, 2, 4\}$
$E_2 = \{2, 3\}$  $E'_2 = \{2, 3, 5\}$
$E_3 = \{3, 4\}$  $E'_3 = \{3, 4\}$
$E_4 = \{1, 4\}$  $E'_4 = \{1, 3, 4, 5\}$
We typically give our results using matrices. Define a matrix to be **simple** if it is a \((0,1)\)-matrix with no repeated columns. A \(k \times \ell\) \((0,1)\)-matrix corresponds to a hypergraph (or set system) of \(\ell\) edges on a ground set of \(k\) vertices where each column is viewed as the incidence matrix of an edge.

\[
F = \begin{bmatrix}
E_1 & E_2 & E_3 & E_4 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} \quad \Leftrightarrow \quad H = \begin{bmatrix}
E'_1 & E'_3 & E'_4 & E'_2 & \cdots \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
Consider a (0,1)-matrix $F$. We say that $A$ has $F$ as a **Berge Hypergraph** if there is a submatrix $B$ of $A$ and a row and column permutation $G$ of $F$ so that $G \leq B$. row/column order doesn’t matter, 0’s don’t matter.
Consider a (0,1)-matrix $F$. We say that $A$ has $F$ as a Berge Hypergraph if there is a submatrix $B$ of $A$ and a row and column permutation $G$ of $F$ so that $G \leq B$.

row/column order doesn’t matter, 0’s don’t matter.

We say that $A$ has $F$ as a Pattern if there is a submatrix $B$ of $A$ so that $F \leq B$.

row/column order matters, 0’s don’t matter.
Consider a $(0,1)$-matrix $F$. We say that $A$ has $F$ as a **Berge Hypergraph** if there is a submatrix $B$ of $A$ and a row and column permutation $G$ of $F$ so that $G \preceq B$.

row/column order doesn’t matter, 0’s don’t matter.

We say that $A$ has $F$ as a **Pattern** if there is a submatrix $B$ of $A$ so that $F \preceq B$.

row/column order matters, 0’s don’t matter.

We say that $A$ has $F$ as a **Configuration** if there is a submatrix $B$ of $A$ and a row and column permutation $G$ of $F$ so that $G = B$.

row/column order doesn’t matter, 0’s matter.
Define $\|A\|$ as the number of columns of $A$. Define our extremal problem as follows:

$$\text{Avoid}(m, F) = \{ A : A \text{ is } m\text{-rowed, simple, } F \not\preceq A \},$$

$$\text{Bh}(m, F) = \max_A \{ \|A\| : A \in \text{Avoid}(m, F) \}.$$
Theorem $\text{Bh}(m, l_k) = 2^{k-1}$

The fact that this is a constant would follow from a result of Balogh and Bollobás (2005). This exact bound follows by induction or by the shifting argument given later.
Extermal Graph Theory

\[ \text{ex}(m, G) \] is the maximum number of edges in a graph on \( m \) vertices which has no subgraph \( G \).
Graph Theory and Berge Hypergraphs

Given a $k \times \ell$ (0,1)-matrix $F$, we can form a graph $G(F)$ on $k$ vertices where we join $i, j$ if there is a column in $F$ with 1’s in rows $i, j$. Alternatively replace the hyperedges in the hypergraph associated with $F$, by the cliques associated with each hyperedge and take the union of the edges.

**Theorem** $\text{Bh}(m, F) \geq \text{ex}(m, G(F)) + m + 1$

**e.g.**

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

has $G(F) = C_4$.

Since $\text{ex}(m, C_4) = \Theta(m^{3/2})$ then $\text{Bh}(m, F)$ is $\Omega(m^{3/2})$. 
$C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

**Theorem** (A., Koch, Raggi, Sali '14) $\text{forb}(m, \{C_4, T_4\})$ is $\Theta(m^{3/2})$.

**Corollary** $\text{Bh}(m, C_4)$ is $\Theta(m^{3/2})$.

**Proof:** $C_4 \ll T_4$ so avoiding $C_4$ as a Berge hypergraph will forbid both $C_4$ and $T_4$ as configurations (as well as some other configurations).
**Theorem** If \( A \in \text{Avoid}(m, F) \), then there exists an \( A' \in \text{Avoid}(m, F) \) with \( \|A\| = \|A'\| \) and the columns of \( A' \) form a downset: namely if \( \alpha \) is a column of \( A' \) and \( \beta \leq \alpha \), then \( \beta \) is a column of \( A' \).

**Proof:** Apply a shifting argument, replacing 1’s by 0’s in \( A \) as long as no repeated columns are created. The result is \( A' \).
Definition The product $I_2 \times I_4$

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
Forbidden Berge Hypergraph $I_2 \times I_4$

**Definition** The product $I_2 \times I_4$

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \times \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

Note that $G(I_2 \times I_4) = K_{2,4}$.
Assume $A \in \text{Avoid}(m, I_2 \times I_4)$ with
\[ A = \begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & \vdots \\
\end{array} \]

Recall that $\text{Bh}(m, I_4) = 2^3$. 

Hence if $I_2 \times I_4 \not\subseteq A$ then for each pair of rows $i, j$, the number of columns of $A$ with 1's on both rows $i, j$ is at most $2^3$. Then the number of columns with three or more 1's is asymptotic to the number of columns of sum 2.
Assume $A \in \mathcal{A}(m, I_2 \times I_4)$ with $A = i j > 2^4 - 1$

Thus $I_2 \times I_4 \preceq A$ using the idea that $A$ is a downset. Hence if $I_2 \times I_4 \preceq A$ then for each pair of rows $i, j$, the number of columns of $A$ with 1's on both rows $i, j$ is at most $2^3$. Then the number of columns with three or more 1's is asymptotic to the number of columns of sum 2.
Definition \( \text{ex}(m, K_\ell, G) \) is the maximum number of copies of \( K_\ell \) in an \( m \)-vertex \( G \)-free graph.

Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman ’15 and Kostachka, Mubayi and Verstratte ’15.
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**Theorem** (Alon, Shikhelman ’15, Kostachka, et al ’15)
Let \( s, t \) be given with \( t \geq (s - 1)! + 1 \). Then
\[
\text{ex}(m, K_3, K_{s,t}) \) is \( \Theta(m^{3-(3/s)}) \).

**Theorem** (Alon, Shikhelman ’15, Kostachka, et al ’15)
Let \( r, s, t \) be given with \( s \geq 2r - 2 \), \( t \geq (s - 1)! + 1 \)
\[
\text{ex}(m, K_r, K_{s,t}) \geq \left( \frac{1}{r!} + o(1) \right) m^{r-\frac{r(r-1)}{2s}}
\]
Definition \( \text{ex}(m, K_{\ell}, G) \) is the maximum number of copies of \( K_{\ell} \) in an \( m \)-vertex \( G \)-free graph. Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman ’15 and Kostachka, Mubayi and Verstratte ’15.
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Lemma Given \( A \in \text{Avoid}(m, I_3 \times I_k) \), where \( A \) is a downset, the number of columns of column sum \( \ell \) (\( \ell \geq 3 \)) in \( A \) is at most \( \text{ex}(m, K_\ell, K_3, k) \).

Theorem \( \text{Bh}(m, I_3 \times I_k) \leq 1 + m + \text{ex}(m, K_3, k) + 2^{k-1}\text{ex}(m, K_3, K_3, k) \)
**Definition** \( \text{ex}(m, K_\ell, G) \) is the maximum number of copies of \( K_\ell \) in an \( m \)-vertex \( G \)-free graph. Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman ’15 and Kostachka, Mubayi and Verstratte ’15.

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**Theorem** \( \text{Bh}(m, I_4 \times I_4) \leq 1 + m + \text{ex}(m, K_4, k) + \text{ex}(m, K_3, K_4, k) + 2^{k-1}\text{ex}(m, K_4, K_4, k) \)
Let $T_k$ be a tree on $k$ vertices. A well known result for trees is $\text{ex}(m, T_k)$ is $\Theta(m)$.

**Theorem** Let $T_k$ be a tree on $k$ vertices and let $F$ be the $k$-rowed vertex-edge incidence matrix of $T_k$ so $G(F) = T_k$ and $F$ has column sums 2. Then $\text{Bh}(m, F)$ is $\Theta(m)$. 
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**Theorem** Let $T_k$ be a tree on $k$ vertices and let $F$ be the $k$-rowed vertex-edge incidence matrix of $T_k$ so $G(F) = T_k$ and $F$ has column sums 2. Then $\text{ Bh}(m, F)$ is $\Theta(m)$.

The situation is different for configurations

**Theorem** Let $T_k$ be a tree on $k$ vertices and let $F$ be the $k$-rowed vertex-edge incidence matrix of $T_k$. Then $\text{forb}(m, F)$ is $\Theta(m^{k-1})$ or $\Theta(m^{k-2})$ or $\Theta(m^{k-3})$ depending on $T_k$. 
Smallest Open Problem

\[ C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \]

where \( G(F) \) is \( C_4 \).

\[ \text{Bh}(m, C_4) \text{ is } \Theta(m^{3/2}). \]

\[ F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

**Problem** What is \( \text{Bh}(m, F) \)?

We might guess that \( \text{Bh}(m, F) \text{ is } \Theta(m^2) \).
THANKS to those who have kept contributing to the CanaDAM series of conference. Another great event!