1. [25 marks] A game is played on a $5 \times 5$ grid. (drawn twice below). A token can be moved either one step to the left, one step down, or one step diagonally left-down. (a) Find the P and N positions.
(b) Find the Sprague-Grundy value of each position.
(c) If we play three copies of the game at once (a sum of the games), starting with positions $a, b, c$, find all winning for player 1, or show there aren’t any.
(d) If the same game is played on an $n \times n$ grid, what are the P-positions? (prove your claims)

\[
\begin{array}{ccccc}
0 & 1 & 0 & 1 & 0 \\
1 & 2 & 1 & 2 & 0 \\
2 & 3 & 2 & 3 & 0 \\
3 & 4 & 3 & 4 & 0 \\
4 & 5 & 4 & 5 & 0 \\
\end{array}
\]

Solution: (a,d) P-positions are $(n, m)$ with both coordinates even (where $(0, 0)$ is the lower left corner.) If both are even then any move makes one of the coordinates odd. If at least one is odd, we can make them both even.
(b) Computed in the grid above.
(c) Move any one of the three diagonally. (explain!)

2. [18 marks] In a splitting game, the players are given a few non-empty piles of stones. A legal move consists of splitting any one pile into two non-empty piles. The winner makes the last move. Thus the terminal positions consist only of piles of size one.
Find and prove an expression for the Sprague-Grundy value of the game for a single pile of any size. Use this to determine for which starting positions (with any number of piles) the first player will win. Justify all claims.

Solution: An even pile has value 1; an odd pile has value 0.
Proof is by induction or verification: An odd pile always turns to an odd and an even pile, with value $0 \oplus 1$, so indeed has value 0. An even pile
split into two odd or two even piles, with values \(0 \oplus 0\) or \(1 \oplus 1 = 0\), so the mex is 1.

If there are several piles the nim-sum of their sizes is 0 (resp. 1) if there is an even (resp. odd) number of even piles. Player 1 wins if there is an odd number of even piles.

3. 12 marks Find the value and all optimal strategies for each player in the zero-sum game with matrix \(A\) below.

\[
A = \begin{pmatrix}
0 & 1 & 2 & 8 \\
6 & 4 & 2 & 0 \\
\end{pmatrix}
\]

Solution: If Player 1 uses \(x = (p, 1-p)\), the minimal outcome is \(\min(6-6p, 4-3p, 2, 8p)\). This is maximized for \(p \in \left[\frac{1}{4}, \frac{2}{3}\right]\) and the value is 2. The only optimal strategy for player 2 is to take column 3. Any other strategy will let player 1 achieve a larger result.

4. 12 marks Find the value and some optimal strategy for each player in the zero-sum game with matrix \(B\) below.

\[
B = \begin{pmatrix}
1 & 0 & 0 & -3 \\
1 & 0 & 0 & 2 \\
0 & 2 & 0 & 1 \\
0 & 0 & 3 & 1 \\
0 & -1 & -2 & 2 \\
\end{pmatrix}
\]

Solution: Rows 1 and 5 are dominated by row 2. After removing them, column 4 is dominated by column 1. This leaves the smaller matrix

\[
B' = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3 \\
\end{pmatrix}
\]

Here we can equalize payoffs: If \(x = y = (6/11, 3/11, 2/11)^T\) then \(x^TB'\) and \(B'y\) are constant vectors. Since \(x, y\) are distributions these are the strategies for \(B'\) and the value is 6/11.

For \(B\) this give \(x = (0, 6/11, 3/11, 2/11, 0)\) and \(y = (6/11, 3/11, 2/11, 0, 0)\).
5. \textbf{15 marks} Find the value and an optimal strategy for the game \( A = \begin{pmatrix} s & 2 \\ 0 & s \end{pmatrix} \) as a function of \( s \). Draw a graph of the value as a function of \( s \) (from \(-5\) to \(5\)).

\textbf{Solution:} If \( 0 \leq s \leq 2 \) then there is a saddle point and the value is \( s \).

Otherwise, we must use mixed strategies. Equalizing payoffs gives \( v = \frac{s^2}{2s - 2} \). This is asymptotically \( s/2 \) as \( s \to \pm\infty \).

6. \textbf{18 marks} Ruth and Chris play the following game. Chris picks a number in \( \{1, 2, 3\} \) and Ruth guesses the number. If she guesses correctly, Chris pays her $2. If her guess is too small Chris pays her $1. If her guess is too large, Chris pays nothing.

(a) Write this game in matrix form.
(b) Find its value and an optimal strategies for each players.
(c) [bonus] What is the value if they can select any number in \( \{1, 2, \ldots, n\} \)?

\textbf{Solution:} (a) \( A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \).

(b) Equalizing payoffs gives \( x = (4/7, 2/7, 1/7)^T \) and \( x^T A = (8/7, 8/7, 8/7) \).

Similarly \( y = (1/7, 2/7, 4/7)^T \) and \( Ay \) is the same constant vector. Since \( x, y \) are probability vectors, these are the optimal strategies.

(c) \( x_i = \frac{2^{n-1}}{2^n - 1} \) and \( y_i = \frac{2^i}{2^n - 1} \), gives value \( \frac{2^n}{2^n - 1} \). Note that if \( n \) is large, Ruth tends to pick very small numbers and Chris very large numbers, and the result is close to 1.