Allocation (fair)

Problem: split a given resource between players.

Fair allocation: each player can be sure to get a fair share.

Each player has a measure $\mu_i$. For a piece $A$ $\mu_i(A)$ is $i$th player value of $A$.

Assume $\mu_i(\Omega) = 1$ for every $i$.

2-player method: (cut and choose):

Player 1 splits $\Omega = A \cup B$ s.t. $\mu_i(A) = \frac{1}{2}$.

Player 2 selects $A$ or $B$.

Player 1 gets other.
Claim: This is fair: every player can guarantee to get $\geq \frac{1}{2}$

If Allocation is $A_1, A_2$ then $\mu(A_i) \geq \frac{1}{2}$ for $i=1, 2$

Pf: p2: take $A$ if $\mu(A) \geq \frac{1}{2}$, otherwise $B$.

p1: cut st. $\mu(A) = \mu(B) = \frac{1}{2}$.

Valid if $\mu$, non-atomic (continuous).

Sliding Knife for $n$ players:

A proposed slice $S_t$ increases continuously.
Each player can say 'stop' once, get the slice $S_t$
Allocate remaining cake to other players.
Claim: this is fair.

Let: strategy for each player is same: once $\mu_i(S_t) = \frac{1}{n}$ say cut.

If I say step first, $\mu_i(A) = \frac{1}{n}$.

If another player does, then $\mu_i(\Omega \setminus S_t) > 1 - \frac{1}{n}$.

By induction, I get at least $\frac{1}{n-1}$ of $\Omega \setminus S$

$$\frac{1}{n-1} \mu_i(\Omega \setminus S) \geq \frac{1}{n-1} \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$
Envy free allocation:

Allocation s.t. \( \mu_i(A_i) \geq \mu_i(A_j) \) \( \forall i, j \)

every player considers his piece to be largest.

For \( n=2 \) envy free \( \iff \) fair.

For \( n>2 \), sliding knife does not give envy free allocation in general.

Thm: there exists an envy free allocation.

If the cake is \([0,1]\), can cut the cake into at most \( n^2 \) pieces.

\begin{align*}
1 & 2 & 1 & 3 & 1 & 2 & 1 \\
\end{align*}

 Conj: Can use \( \leq 2n \) cuts
Allocating money.

3 creditors claim 100, 200, 300 owed to them.

How do you allocate estate of size $S < 600$